Trigonometry

Carl W. Lee
University of Kentucky

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1 Radians

Suppose you have a circle of radius 1. Its circumference is $C = 2\pi r = 2\pi$, which is a bit bigger than 6.2.

Problem 1.1 Explain why the formula for the circumference of a circle provides the *definition* of π.

The measure of a central angle that cuts off a piece (intercepts an arc) of the circumference of length 1 is called a *radian*. In general, the measure of an angle that intercepts an arc of the circumference having length ℓ is said to have measure ℓ radians. Therefore, there are 2π radians around the center of a circle and we can convert back and forth between degrees and radians by

$$\theta(\text{in radians}) = \frac{\pi}{180^\circ}\theta(\text{in degrees})$$

$$\theta(\text{in degrees}) = \frac{180^\circ}{\pi}\theta(\text{in radians})$$

Using radians makes many formulas look "nicer." For example, Suppose C is a circle of radius r. The length ℓ of an arc intercepted by a central angle θ is given by

$$\boxed{\begin{aligned} \ell &= r\theta \quad \text{(if } \theta \text{ is measured in radians)} \\ \ell &= \frac{\pi}{180^\circ} r\theta \quad \text{(if } \theta \text{ is measured in degrees)} \end{aligned}}$$

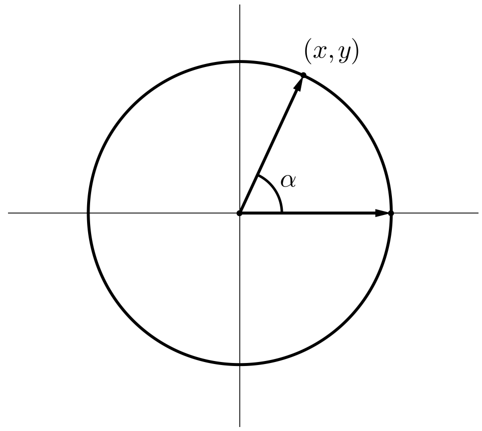
So the radian measure of the central angle is the ratio of the length of the arc and the radius.

Problem 1.2 Propose an analogous definition of the measure of a solid angle where three, four, or more planes meet at common vertex of a polyhedron, and explain why your definition is reasonable. Then look up the official name and definition of solid angle measure.

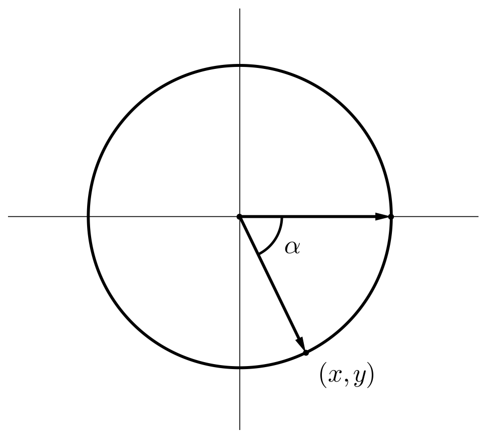
2 Trigonometric Functions in the Analytic Model $\mathbf{E}^2$

A circle of radius one is called a *unit circle*. A unit circle with center at the origin of the Cartesian plane is often called *the unit circle*. The trigonometric functions sine, cosine, tangent, secant, cosecant, and cotangent, can be defined using the unit circle.

Let α be the radian measure of an angle. Place a ray r from the origin along the x axis. If $\alpha \geq 0$, rotate the ray by α radians in the counterclockwise direction.



If $\alpha < 0$, rotate the ray by $|\alpha|$ radians in the clockwise direction.



Determine the point (x, y) where r intersects the unit circle. We define

$$\cos\alpha = x$$

and

$$\sin\alpha = y.$$

Define also

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}.$$

$$\sec\alpha = \frac{1}{\cos\alpha},$$

$$\csc\alpha = \frac{1}{\sin\alpha},$$
$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

Problem 2.1 Use the definitions for the sine, cosine, and tangent functions to evaluate $\sin\alpha$, $\cos\alpha$ and $\tan\alpha$ when α equals

1. 0
2. $\frac{\pi}{2}$
3. π
4. $\frac{3\pi}{2}$
5. 2π
6. $\frac{\pi}{3}$
7. $\frac{\pi}{4}$
8. $\frac{\pi}{6}$
9. $\frac{n\pi}{3}$ for all possible integer values of n
10. $\frac{n\pi}{4}$ for all possible integer values of n
11. $\frac{n\pi}{6}$ for all possible integer values of n

Problem 2.2 Drawing on the definitions for the sine and cosine functions, sketch the graphs of the functions $f(\alpha) = \sin\alpha$ and $f(\alpha) = \cos\alpha$, and explain how you can deduce these naturally from the unit circle definition,

Problem 2.3 Continuing to think about the unit circle definition, complete the following formulas and give brief explanations, including a diagram, for each.

1. $\sin(-\alpha) = -\sin(\alpha)$.

2. $\cos(-\alpha) =$
3. $\sin(\pi + \alpha) =$
4. $\cos(\pi + \alpha) =$
5. $\sin(\pi - \alpha) =$
6. $\cos(\pi - \alpha) =$
7. $\sin(\pi/2 + \alpha) =$
8. $\cos(\pi/2 + \alpha) =$
9. $\sin(\pi/2 - \alpha) =$
10. $\cos(\pi/2 - \alpha) =$
11. $\sin^2(\alpha) + \cos^2(\alpha) =$

Solution. (Diagrams omitted.)

1. $\sin(-\alpha) = -\sin(\alpha)$.
2. $\cos(-\alpha) = \cos\alpha$.
3. $\sin(\pi + \alpha) = -\sin(\alpha)$.
4. $\cos(\pi + \alpha) = -\cos(\alpha)$.
5. $\sin(\pi - \alpha) = \sin(\alpha)$.
6. $\cos(\pi - \alpha) = -\cos(\alpha)$.
7. $\sin(\pi/2 + \alpha) = \cos(\alpha)$.
8. $\cos(\pi/2 + \alpha) = -\sin(\alpha)$.
9. $\sin(\pi/2 - \alpha) = \cos(\alpha)$.
10. $\cos(\pi/2 - \alpha) = \sin(\alpha)$.
11. $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

Problem 2.4 Use GeoGebra to make a sketch of the unit circle to illustrate what you have learned so far.

Problem 2.5 Use the sine and cosine functions to determine the coordinates of the vertices of the following. In each case except the last two, choose one vertex to be the point $(1, 0)$.

1. A regular triangle with vertices having a distance of 1 from the origin.

   Solution. $(1, 0)$, $(\cos 120°, \sin 120°) = (-1/2, \sqrt{3}/2)$, $(\cos 240°, \sin 240°) = (-1/2, -\sqrt{3}/2)$.

2. A regular square with vertices having a distance of 1 from the origin.

3. A regular pentagon with vertices having a distance of 1 from the origin.

   Solution. $(1, 0)$, $(\cos 72°, \sin 72°)$, $(\cos 144°, \sin 144°)$, $(\cos 216°, \sin 216°)$, $(\cos 288°, \sin 288°)$.

4. A regular hexagon with vertices having a distance of 1 from the origin.

   Solution. $(1, 0)$, $(\cos 60°, \sin 60°) = (1/2, \sqrt{3}/2)$, $(\cos 120°, \sin 120°) = (-1/2, \sqrt{3}/2)$, $(\cos 180°, \sin 180°) = (-1, 0)$, $(\cos 240°, \sin 240°) = (-1/2, -\sqrt{3}/2)$, $(\cos 300°, \sin 300°) = (1/2, -\sqrt{3}/2)$.

5. A regular heptagon with vertices having a distance of 3 from the origin.

6. A regular n-gon with vertices having a distance of r from the origin.

Problem 2.6 Confirm the above calculations by entering the coordinates of the above points into GeoGebra.

Problem 2.7 Here is perhaps a more familiar way to define sine and cosine for an acute angle α: Take any right triangle for which one of the angles measures α. Then $\sin \alpha$ is the ratio of the lengths of the opposite side and the hypotenuse, and $\cos \alpha$ is the ratio of the lengths of the adjacent side and the hypotenuse. Explain why this definition gives the same result as the unit circle.

Problem 2.8 Describe a procedure to determine the rectangular coordinates (x, y) of a point from its polar coordinates (r, θ) and justify why it works.

Problem 2.9 Look up the definitions of cylindrical and spherical coordinates.

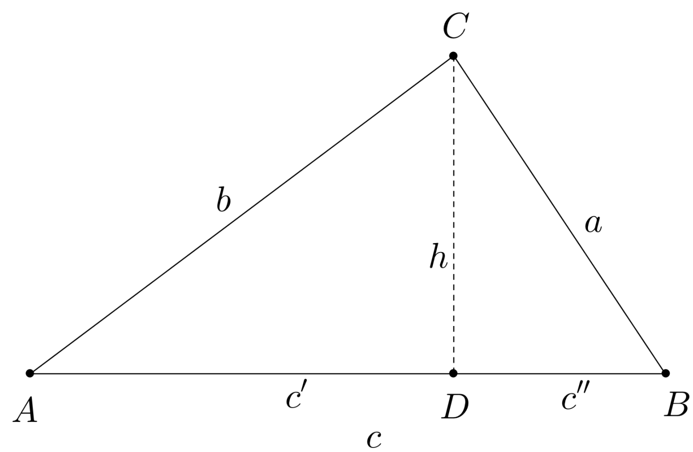
1. Justify the following conversion from cylindrical coordinates (r, θ, z) to rectangular coordinates (x, y, z).
$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

2. Justify the following conversion from spherical coordinates (r, θ, ϕ) to rectangular coordinates (x, y, z).
$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned}$$

3 Trigonometric Identities

Problem 3.1 In this problem we will use the triangle pictured below. In this triangle all angles have measure less than 90°; however, the results hold true for general triangles.

The lengths of $\overline{BC}$, $\overline{AC}$ and $\overline{AB}$ are a, b and c, respectively. Segment $\overline{AD}$ has length c' and $\overline{DB}$ length c''. Segment $\overline{CD}$ is the altitude of the triangle from C, and has length h.



The usual formula for the area of a triangle is $\frac{1}{2}$(base)(height), as you probably already know.

1. Prove that $\text{area}\,(\triangle ABC) = \frac{1}{2}bc\sin A$.

   Solution. $\sin A = \frac{h}{b}$, so $h = b\sin A$. $\text{area}\,\triangle ABC = \frac{1}{2}hc = \frac{1}{2}bc\sin A$.

2. What is a formula for $\text{area}\,(\triangle ABC)$ using $\sin B$? Using $\sin C$? (Note: you will have to use the altitude from A or B).

   Solution.
   $$\text{area}\,(\triangle ABC) = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$

3. What is the relationship of these formulas to the SAS triangle congruence criterion?

   Solution. Since the triangle is determined up to congruence by SAS, then so is its area, and the formulas show exactly how—the area is half of the product of the two sides and the sine of the included angle.

Problem 3.2 Using the same triangle, the *Law of Sines* is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

1. We showed that the area of this triangle was given by three different formulas. What are they?

   Solution.
   $$\text{area}\,(\triangle ABC) = \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C.$$

2. From these three formulas, prove the Law of Sines.

   Solution. Divide the above equations through by $\frac{1}{2}abc$.

Problem 3.3 The *Law of Cosines* is:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Using the above triangle:

1. Show that $c' = b \cos A$.

   Solution. Observe that $\cos A = c'/b$ from triangle $\triangle ACD$.

2. Observe the obvious fact that $c'' = c - c'$.

   Solution. This is clear from the diagram.

3. Verify that $h^2 = b^2 - (c')^2$.

   Solution. This follows from the Pythagorean Theorem applied to triangle $\triangle ACD$.

4. Apply the Pythagorean Theorem to triangle $\triangle CDB$, then use the facts above to make the appropriate substitutions to prove the Law of Cosines.

   Solution.
$$\begin{aligned} a^2 &= h^2 + (c'')^2 \\ &= b^2 - (c')^2 + (c - c')^2 \\ &= b^2 - (c')^2 + c^2 - 2cc' + (c')^2 \\ &= b^2 + c^2 - 2cc' \\ &= b^2 + c^2 - 2bc\cos A. \end{aligned}$$

5. What happens when you apply the Law of Cosines in the case that $\angle A$ is a right angle?

   Solution. In this case $\cos A = 0$ and you get $a^2 = b^2 + c^2$, which is the Pythagorean Theorem.

Problem 3.4 Suppose for a triangle you are given the lengths of the three sides. How can you determine the measures of the three angles?

Problem 3.5 Suppose for a triangle you are given the lengths of two sides and the measure of the included angle. How can you determine the length of the other side, and the measures of the other two angles?

Problem 3.6 Suppose for a triangle you are given the measures of two angles and the length of the included side. How can you determine the measure of the other angle, and the lengths of the other two sides?

Problem 3.7 Assume that you have triangle $\triangle ABC$ such that the coordinates of the three (distinct) points A, B, and C are $(0,0)$, (x_1, y_1), and (x_2, y_2), respectively. Use the Law of Cosines and the distance formula to prove that

$$\boxed{\cos A = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.}$$

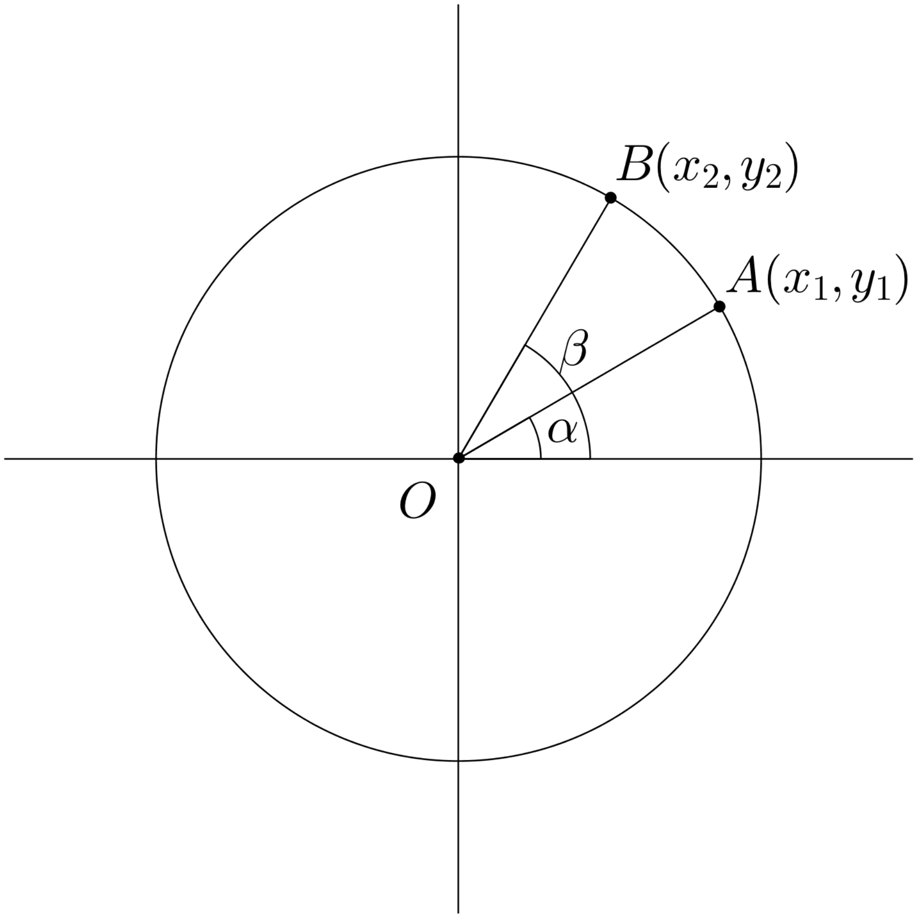
Solution. Beginning with $a^2 = b^2 + c^2 - 2bc\cos A$, solve for $\cos A$.

$$\begin{aligned}
\cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\
&= \frac{(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - [(x_2 - x_1)^2 + (y_2 - y_1)^2]}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{x_1^2 + y_1^2 + x_2^2 + y_2^2 - (x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2)}{2\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}} \\
&= \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}.
\end{aligned}$$

It is an easy extension to determine the angle formed by three points when the vertex of the angle is not the origin. Assume that you have triangle $\triangle ABC$ such that the coordinates of the three (distinct) points A, B, and C are (x_3, y_3), (x_1, y_1), and (x_2, y_2), respectively. Let $x_1' = x_1 - x_3$, $x_2' = x_2 - x_3$, $y_1' = y_1 - y_3$, and $y_2' = y_2 - x_3$. Then

$$\boxed{\cos A = \frac{x_1'x_2' + y_1'y_2'}{\sqrt{x_1'^2 + y_1'^2}\sqrt{x_2'^2 + y_2'^2}}.}$$

Problem 3.8 Assume that A and B are two points on the unit circle centered at the origin, with respective coordinates (x_1, y_1) and (x_2, y_2). Draw the line segments $\overline{OA}$ and $\overline{OB}$. Let α be the angle that $\overline{OA}$ makes with the positive x-axis, and β be the angle that $\overline{OB}$ makes with the positive x-axis.



1. From Problem 3.7 we know that

$$\cos(\angle AOB) = \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}}$$

From this, prove that

$$\boxed{\cos(\beta - \alpha) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$$

Solution. Use the fact that $(x_1, y_1) = (\cos\alpha, \sin\alpha)$ and $x_1^2 + y_1^2 = 1$, and the analogous facts about β.

2. Replace α with $-\alpha$ in the previous equation to prove

$$\boxed{\cos(\beta + \alpha) = \cos\beta\cos\alpha - \sin\beta\sin\alpha.}$$

Solution. $\cos(\beta+\alpha) = \cos(\beta-(-\alpha)) = \cos(-\alpha)\cos\beta + \sin(-\alpha)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

3. Replace β with $\pi/2 - \gamma$ and α with $-\delta$ in the previous equation to prove

$$\boxed{\sin(\gamma + \delta) = \sin\gamma\cos\delta + \cos\gamma\sin\delta.}$$

Solution.

$$\begin{aligned}
\cos(\beta + \alpha) &= \cos\beta\cos\alpha - \sin\beta\sin\alpha \\
\cos(\pi/2 - \gamma - \delta) &= \cos(\pi/2 - \gamma)\cos(-\delta) - \sin(\pi/2 - \gamma)\sin(-\delta) \\
\sin(\gamma + \delta) &= \sin\gamma\cos\delta + \cos\gamma\sin\delta
\end{aligned}$$

4. Replace δ with $-\delta$ in the previous equation to prove

$$\boxed{\sin(\gamma - \delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta.}$$

Solution. $\sin(\gamma - \delta) = \sin(\gamma + (-\delta)) = \sin\gamma\cos(-\delta) + \cos\gamma\sin(-\delta) = \sin\gamma\cos\delta - \cos\gamma\sin\delta$.

The above four formulas are the trigonometric *angle sum* and *angle difference formulas.*

Problem 3.9 Prove the *double angle* formulas:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha.$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha.$$

Solution. Set $\gamma = \alpha$ and $\delta = \alpha$ in the formula for $\sin(\gamma + \delta)$, and set $\beta = \alpha$ in the formula for $\cos(\alpha + \beta)$. Then use $\sin^2\alpha + \cos^2\alpha = 1$.

Problem 3.10 Prove the *half angle* formulas for angle $0 \le \beta \le \frac{\pi}{2}$.

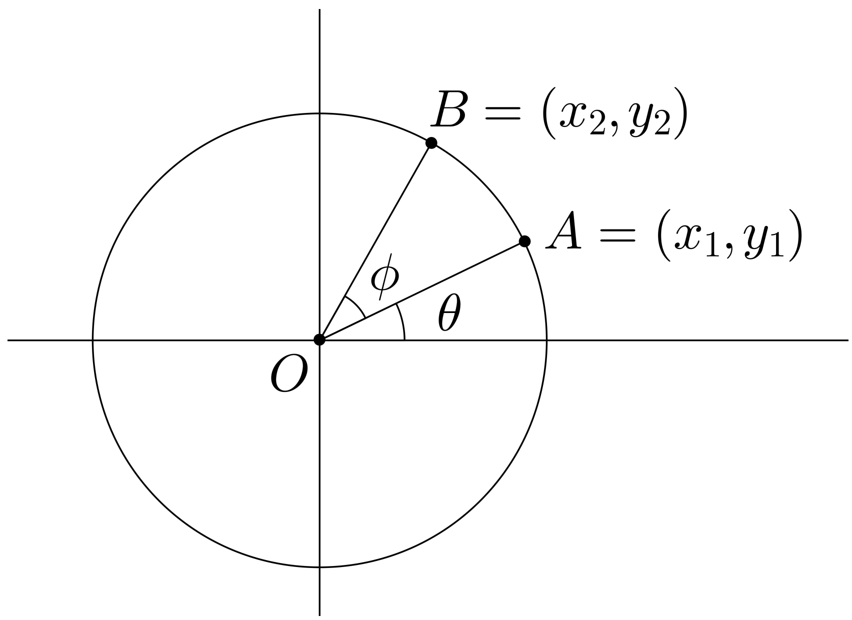
$$\sin(\beta/2) = \sqrt{\frac{1-\cos\beta}{2}}.$$

$$\cos(\beta/2) = \sqrt{\frac{1+\cos\beta}{2}}.$$

Set $\alpha = \frac{\beta}{2}$ in the second two formulas for $\cos(2\alpha)$ and then solve for $\cos(\frac{\beta}{2})$ and $\sin(\frac{\beta}{2})$.

4 Rotations

Problem 4.1 Now assume that we have a circle of radius r, that point A has coordinates $(x_1, y_1) = (r\cos\theta, r\sin\theta)$, and that we wish to rotate it by ϕ about the origin, obtaining the point $B = (x_2, y_2) = (r\cos(\theta+\phi), r\sin(\theta+\phi))$.



1. Prove that
$$(x_2, y_2) = (x_1\cos\phi - y_1\sin\phi, x_1\sin\phi + y_1\cos\phi).$$

   Solution. $x_2 = r\cos(\theta+\phi) = r[\cos\theta\cos\phi - \sin\theta\sin\phi] = r\cos\theta\cos\phi - r\sin\theta\sin\phi = x_1\cos\phi - y_1\sin\phi$.

   $y_2 = r\sin(\theta+\phi) = r[\sin\theta\cos\phi + \cos\theta\sin\phi] = r\sin\theta\cos\phi + r\cos\theta\sin\phi = y_1\cos\phi + x_1\sin\phi$.

2. Conclude that:

The matrix for the rotation centered at the origin by the angle ϕ is

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

where $c = \cos\phi$ and $s = \sin\phi$.

That is to say, prove that

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

Problem 4.2 Find the matrices for the rotations about the origin by each angle ϕ, $0 \le \phi < 360°$, that is a multiple of 90 degrees.

Problem 4.3 Find the matrices for the rotations about the origin by each angle ϕ, $0 \le \phi < 360°$, that is a multiple of 45 degrees.

Problem 4.4 Find the matrices for the rotations about the origin by each angle ϕ, $0 \le \phi < 360°$, that is a multiple of 30 degrees.

5 Complex Numbers

Any complex number $z = a + bi$ can be represented by a point (a, b) in the Cartesian plane. The real number a is called the *real part* of z, and the real number (not including the i) is called the *imaginary part* of z. But you can set $r = \sqrt{a^2 + b^2}$ and find θ such that $\cos\theta = a/r$ and $\sin\theta = b/r$. That is, (r, θ) are polar coordinates for the point (a, b). Then $z = r(\cos\theta + i\sin\theta)$. The angle θ is called the *argument* of z, denoted $\arg z$, and the length r is called the *modulus* of z, denoted $|z|$. Note: Sometimes $r(\cos\theta + i\sin\theta)$ is written $r\text{cis}\,\theta$.

Problem 5.1 Suppose $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, corresponding to the points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, respectively, in the Cartesian plane. Explain how to find $z = z_1 + z_2$ geometrically. Explain how to find $z = z_1 - z_2$ geometrically.

Solution (for addition): Draw segments $\overline{OP_1}$ and $\overline{OP_2}$, and "complete the parallelogram," The fourth point of the parallelogram will be the point for $z_1 + z_2$.

Another way of viewing this is to think of $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ as directed segments representing vectors. Add them by placing the tail of $\overrightarrow{OP_2}$ at the head of $\overrightarrow{OP_1}$.

For the subtraction $\overrightarrow{OP_1} - \overrightarrow{OP_2}$, add the negative of $\overrightarrow{OP_2}$.

> To add two complex numbers $P_1 = x_1 + iy_1$ and $P_2 = x_2 + iy_2$ geometrically, draw segments $\overline{OP_1}$ and $\overline{OP_2}$ and complete the parallelogram.
>
> To subtract two complex numbers $P_1 = x_1 + iy_1$ and $P_2 = x_2 + iy_2$ geometrically, let $P_3 = -P_2$, draw segments $\overline{OP_1}$ and $\overline{OP_3}$, and complete the parallelogram.

Problem 5.2 Suppose $z_1 = r_1\text{cis}\,\theta_1$ and $z_2 = r_2\text{cis}\,\theta_2$, corresponding to the points P_1, P_2 in the Cartesian plane with polar coordinates (r_1, θ_1), (r_2, θ_2), respectively. Explain how to find $z = z_1 z_2$ geometrically. Explain how to find $z = z_1/z_2$ geometrically.

Solution: Preliminary to this, it might be helpful to think about multiplication of real numbers in the following way: Think of the x-axis in the plane, and real numbers represented by segments from the origin to points on this number line. When multiplying by a negative number, rotate 180° first, then multiply (scale) by the absolute value.

Use polar coordinates to write $z_1 = r_1 \cos\theta_1 + ir_1 \sin\theta_1$ and $z_2 = r_2 \cos\theta_2 + ir_1 \sin\theta_2$. Calculate

$$\begin{aligned} z_1 z_2 &= (r_1 \cos\theta_1 + ir_1 \sin\theta_1)(r_2 \cos\theta_2 + ir_2 \sin\theta_2) \\ &= r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)) \\ &= r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)). \end{aligned}$$

Interpretation: Draw segments $\overline{OP_1}$ and $\overline{OP_2}$. So you need to construct a segment $\overline{OP}$ such that its angle with respect to the positive x-axis is the sum $\theta_1 + \theta_2$ of the angles associated with $\overline{OP_1}$ and $\overline{OP_2}$, and the distance of P from O is the product $r_1 r_2$ of the distances OP_1 and OP_2.

> To multiply two complex numbers $P_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $P_2 = r_2(\cos\theta_2 + i \sin\theta_2)$ geometrically, add the angles (arguments) $\theta_1 + \theta_2$ and multiply the lengths (moduli) $r_1 r_2$.
>
> To divide two complex numbers $P_1 = r_1(\cos\theta_1 + i \sin\theta_1)$ and $P_2 = r_2(\cos\theta_2 + i \sin\theta_2)$ geometrically, subtract the angles (arguments) $\theta_1 - \theta_2$ and divide the lengths (moduli) r_1 / r_2.

Problem 5.3 Prove the following:

> Let $w = r(\cos\phi + i \sin\phi)$. Then the function $f(z) = wz$ rotates the complex plane counterclockwise by the angle ϕ and then scales it by a factor of r.

Problem 5.4 From what you learned in the previous exercise,

1. Explain geometrically what multiplying by i does.
2. Show geometrically that $i^2 = -1$.

Solution: The point $i = 0 + 1i$ corresponds to the point P with Cartesian coordinates $(1, 0)$. The angle of $\overline{OP}$ with respect to the positive x-axis is $\pi/2$, and the distance OP is 1. Multiplying i by itself thus results in a point Q such that the angle of $\overline{OQ}$ with respect to the positive x-axis is π, and the distance OQ is 1. So $Q = (-1, 0)$, which corresponds to the complex number $-1 + 0i = -1$.

3. Find three complex numbers such that $z^3 = 1$.
4. Find three complex numbers such that $z^3 = 27$.
5. Find four complex numbers such that $z^4 = 1$.
6. Find four complex numbers such that $z^4 = \frac{1}{16}$.
7. Find six complex numbers such that $z^6 = 1$.
8. Find two complex numbers such that $z^2 = i$.
9. Find three complex numbers such that $z^3 = 8i$.
10. Explain how to calculate z^n for any particular complex number z, where n is a positive integer.
11. Explain how to find all solutions to any equation of the form $z^n = z_0$ where n is a positive integer and z_0 is a particular complex number.

Problem 5.5 Show that if we map or identify the complex number $x + iy = r\text{cis}\,\theta$ with the 2×2 matrix

$$\begin{bmatrix} rc & -rs \\ rs & rc \end{bmatrix}, \text{ equivalently, } \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where $c = \cos\theta$ and $s = \sin\theta$, then we can add and multiply complex numbers by simply adding and multiplying their associated matrices. Thus, this set of matrices is a *representation* of, or *isomorphic* to, the complex numbers.

Note also that the subset of matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

is isomorphic to the set of real numbers.

We have seen that radians are a natural unit for getting a nice formula for the length of a circular arc: $\ell = r\theta$ if θ is the central angle measured in radians, r is the radius of the circle, and ℓ is the length of the arc. Another motivation for expressing angles is radians is the Taylor series formulas for sine and cosine:

For angle x measured in radians:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Problem 5.6 Derive these Taylor series.

Solution: Let $f(x) = \sin x$. Then

$$\begin{aligned} & f(0)x^0/0! + f'(0)x/1! + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\ = \; & \sin(0)/0! + \cos(0)x/1! - \sin(0)x^2/2! - \cos(0)x^3/3! + \sin(0)x^4/4! + \cos(0)x^5/5! - \sin(0)x^6/6! - \cdots \\ = \; & \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$

Let $f(x) = \cos x$. Then

$$\begin{aligned} & f(0)x^0/0! + f'(0)x/1! + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\ = \; & \cos(0)/0! - \sin(0)x/1! - \cos(0)x^2/2! + \sin(0)x^3/3! + \cos(0)x^4/4! - \sin(0)x^5/5! - \cos(0)x^6/6! + \cdots \\ = \; & \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Problem 5.7 Sum the squares of the above series to verify that $\sin^2 x + \cos^2 x = 1$.

This might remind you of the Taylor series for e^x:

$$\boxed{e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots}$$

Problem 5.8 Derive this Taylor series.

Solution: Let $f(x) = e^x$. Then

$$\begin{aligned} & f(0)x^0/0! + f'(0)x/1! + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\ = \; & e^0/0! + e^0x/1! + e^0x^2/2! + e^0x^3/3! + e^0x^4/4! + e^0x^5/5! + e^0x^6/6! + \cdots \\ = \; & \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \end{aligned}$$

Problem 5.9 Use the above series to show that $e^a e^b = e^{a+b}$.

From substitution (and some observations about convergence), one gets the beautiful formula for all complex numbers x:

$$\boxed{e^{ix} = \cos x + i \sin x}$$

In particular, setting $x = \pi$ yields an expression containing the perhaps five most important constants in mathematics:

$$\boxed{e^{i\pi} + 1 = 0}$$

These formulas provide a connection between two representations for complex numbers on the one hand, and Cartesian and polar coordinates on the other. Any complex number $r\text{cis}\theta$ can now also be written $re^{i\theta}$. Because $r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$ we have another way to see that to multiply two complex numbers we add the angles (arguments) and multiply the lengths (moduli).

Problem 5.10 Suppose $z = z_1 z_2$ where $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. Use Problem 5.9 to prove the angle sum formulas for sine and cosine.