Discrete Math Notes

1 The Twelve-Fold Way

Count the numbers of ways to place a collection X of $m \geq 1$ balls into a collection Y of $n \geq 1$ boxes, with the following options:

- The balls are either distinguishable (labeled) or indistinguishable (unlabeled)
- The boxes are either distinguishable (labeled) or indistinguishable (unlabeled)
- The placement is either unrestricted, injective (one-to-one), or surjective (onto)

Some questions to think about:

- Can you interpret the problem in another way?
- Is there a closed formula?
- Is there a recursive formula?
- Over what values of m and n do these formulas hold?
- What are some related problems?

1. X: m Labeled Balls, Y: n Labeled Boxes, Unrestricted Placement

   (a) Interpretation

   Counts the number of functions from X into Y. We get a one-to-one matching of ball placements and functions g by placing ball i into box j if and only if $g(i) = j$.

   (b) Closed Formula

   Solved the problem with 3 balls and 5 boxes by considering possible configurations: all three in one box, two in one and one in another, and all three in separate boxes.

   Call the number of placements $f(m, n)$. Proved $f(m, n) = n^m$ using the multiplication counting principle. This principle can be proved by mathematical induction.

   Yields 1 if $m = 0$ and $n > 0$, and 0 if $m > 0$ and $n = 0$, both of which "make sense."

   Motivates the notation of Y^X for the set of functions from X into Y, of which there are $|Y|^{|X|}$.

   (c) Recursive Formula

   $f(m, n) = n$ if $m = 1$; $f(m, n) = n \cdot n^{m-1}$ if $m > 1$. Proved this by induction on m. Wondered if it would be possible to prove this by induction on n.

   Discussed using the recursive formula to make a spreadsheet of values.

   (d) Related Problems

   If $n = 2$ this problem is isomorphic to counting the number of subsets of X, denoted 2^X, of which there are $2^{|X|} = 2^m$. Think of "2" as representing a set with two elements; e.g, $\{0, 1\}$. Discussed a direct inductive proof that the number of subsets of $X = \{1, \ldots, m\}$ is 2^m by looking at all of the subsets $\{1, \ldots, m-1\}$, and then either adding or not adding the element m to each.

   How many ways are there to partition X into n labeled subsets of size $a_1, \ldots, a_n$, respectively, where $a_1 + \cdots + a_n = m$ and $a_1, \ldots, a_n \geq 0$? The answer is the *multinomial coefficient*

$$\binom{m}{a_1, \ldots, a_n} = \frac{m!}{a_1! \cdots a_n!}.$$

   We saw two proofs of this formula. So over all choices of $a_1, \ldots, a_n$ (which are weak compositions of m into n parts; see (4)) these must sum to n^m:

$$\sum_{a_1,\ldots,a_n \in \mathbf{Z},\ a_1,\ldots,a_n \geq 0,\ a_1+\cdots+a_n=m} \binom{m}{a_1, \ldots, a_n} = n^m.$$

Also we discussed the Multinomial Theorem, which states that

$$(x_1 + \cdots + x_n)^m = \sum_{a_1,\ldots,a_n \in \mathbf{Z},\ a_1,\ldots,a_n \geq 0,\ a_1+\cdots+a_n=m} \binom{m}{a_1,\ldots,a_n} x_1^{a_1} \cdots x_n^{a_n}.$$

We started constructing "Pascal's tetrahedron" using the trinomial coefficients.

(e) Additional Comments

Discussed the connection between recursive formulas and recursive programming. For example, $n!$ can be defined to be 1 if $n = 0$ and $n \cdot (n-1)!$ if $n > 0$. Also, the Towers of Hanoi puzzle to move n disks from peg a to peg b using peg c can be solved recursively as:

i. If $n = 1$ then move disk n from a to b.
ii. If $n > 1$ then move $n-1$ disks from a to c using b; move disk n from a to b; move $n-1$ disks from c to b using a.

Discussed the definition of a relation between X and Y as a collection of subsets of $X \times Y$, and of a function from X into Y as a relation between X and Y in which every element of X appears as the first coordinate of exactly one ordered pair. With this definition, the number of functions from X into Y is 1 if X is empty, and 0 if Y is empty but X is not.

Discussed the notion of statements about elements of the empty set being vacuously true, motivated, for example, by the equivalence of an implication and the contrapositive statement. For example, given an integer x, "if x is in the empty set then x is even" is equivalent to "if x is odd then x is not in the empty set."

Another recursive formula: Group the possible placements of balls into boxes according to how many balls are placed into box n. There are $f(m, n-1)$ placements with no balls in box n. There are $\binom{m}{1} f(m-1, n-1)$ placements with exactly one ball in box n—first choose the ball to go into box n, and then place the remaining $m-1$ balls in the remaining $n-1$ boxes. There are $\binom{m}{2} f(m-2, n-1)$ placements with exactly two balls in box n—first choose the two balls to go into box n, and then place the remaining $m-2$ balls in the remaining $n-1$ boxes. And so on. This results in the recursive formula

$$f(m,n) = \binom{m}{0} f(m, n-1) + \binom{m}{1} f(m-1, n-1) + \cdots + \binom{m}{m} f(0, n-1).$$

We saw how to use the recursive formula to find a new proof by induction on n that $f(m, n) = n^m$. This proof used the Binomial Theorem:

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}.$$

2. X: m Labeled Balls, Y: n Labeled Boxes, Injective Placement

   (a) Interpretation

   Counts the number of injective functions from X into Y.

   (b) Closed Formula

   $n(n-1)(n-2)\cdots(n-m+1)$. Denoted $(n)_m$ or $n^{\underline{m}}$, the falling factorial function. This is ${}_nP_m$ on some calculators.

   Can be proved with the multiplication counting principle, or induction from the recursive formula.

   Works even if $m > n$, yielding 0.

   It might be helpful to think of lining up the balls in a fixed location and then dragging the boxes over to the balls.

   (c) Recursive Formula

   $(n)_1 = n$; $(n)_m = n \cdot (n-1)_{m-1}$ for $m > 1$. How can you make a spreadsheet for this?

   (d) Related Problems

   Yields $n!$ if $m = n$, the number of bijections from X to Y, and the number of permutations of X (or of Y).

   (e) Additional Comments

   The *Pigeonhole Principle* essentially states that there are no injections if $m > n$—that for any function from X into Y in such a case, there must be at least one box (pigeonhole) with at least two balls (pigeons).

   We saw how the Pigeonhole Principle can be used to prove: Given any five points in a unit square, there exists at least two of the points that are no more than $\sqrt{2}/2$ apart.

3. X: m Labeled Balls, Y: n Labeled Boxes, Surjective Placement

   (a) Interpretation

   Counts the number of surjective functions from an m-element set into an n-element set. Let's denote this by $\hat{S}(m,n)$.

   We saw several different ways to count these functions in some specific cases.

   (b) Closed Formula

   Uses a counting principle related to the Principle of Inclusion/Exclusion; e.g., $|A\cup B\cup B| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$.

   Count all functions from X into Y, remove those from X into Z for each possible subset Z of Y of cardinality $n-1$, return those from X into Z for each possible subset Z of Y of cardinality $n-2$, etc.

$$\hat{S}(m,n) = \binom{n}{n}n^m - \binom{n}{n-1}(n-1)^m + \binom{n}{n-2}(n-2)^m - \cdots + (-1)^{n-1}\binom{n}{1}1^m.$$

   In summation notation:

$$\hat{S}(m,n) = \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^m.$$

   Why does this work? Think of a function from X into Y that is surjective. Then it is one of the functions counted by the term n^m, and it is not among the functions counted by any of the other terms, so this function contributes 1 to the overall sum.

   Now think of a function f from X into Y that is not surjective. Suppose the range is R, a subset of Y of size $r < n$. The function f is counted once by the term n^m. It is counted once for every Z of cardinality $n-1$ containing R, and there are $\binom{n-r}{n-r-1}$ such Z—we must select the r elements of R to place in Z, and then we must select $n-1-r$ more of the $n-r$ remaining. It is counted once for every Z of cardinality $n-2$ containing R, and there are $\binom{n-r}{n-r-2}$ such Z—we must select the r elements of R to place in Z, and then we must select $n-2-r$ more of the $n-r$ remaining. And so on. So f contributes the following to the sum:

$$1 - \binom{n-r}{n-r-1} + \binom{n-r}{n-r-2} - \cdots \pm \binom{n-r}{0}.$$

   This is an alternating sum of binomial coefficients, which equals zero.

We saw two proofs why $\binom{a}{0} - \binom{a}{1} + \binom{a}{2} - \cdots \pm \binom{a}{a} = 0$: One involving a pairing (bijection) between subsets of $\{1, \ldots, a\}$ of even cardinality and subsets of odd cardinality, and another involving the binomial theorem.

The reasoning shows that the formula must work even if $m < n$, giving a binomial identity since $S(m, n) = 0$ in this case.

Table of some values:

$\hat{S}(m,n)$	$n = 1$	2	3	4	5
$m = 1$	1				
2	1	2			
3	1	6	6		
4	1	14	36	24	
5	1	30	150	240	120

$\hat{S}(m, n) = n!S(m, n)$, where $S(m, n)$ is a Stirling number of the second kind. See (9).

We can invert this formula and count the total number of functions from X to Y by looking at their potential ranges:

$$\begin{aligned} n^m &= \binom{n}{0}\hat{S}(m,n) + \binom{n}{1}\hat{S}(m,n-1) + \cdots + \binom{n}{n-1}\hat{S}(m,1) \\ &= \sum_{i=0}^{n}\binom{n}{i}\hat{S}(m,n-i) \\ &= \sum_{i=0}^{n}\binom{n}{i}\hat{S}(m,i). \end{aligned}$$

(c) Recursive Formula

We can see that $\hat{S}(m, n) = n\hat{S}(m-1, n) + n\hat{S}(m-1, n-1)$ in the following way: Think of surjective placements as ordered piles of balls. For every surjective placement of $m-1$ balls into $n-1$ piles, we can add a single pile containing the mth ball in n possible locations with respect to the existing $n-1$ piles. And for every surjective placement of $m-1$ balls into n piles, we can add the mth ball to any one of the existing n piles, hence, n choices. These placements are reversible by removing the mth ball.

(d) Related Problems

When we look at finite differences, we will see why the above implies that the numbers $\hat{S}(m,n)$ appear as the first diagonal of the finite difference table for the function n^m for fixed m:

n	0		1		2		3		4		5		6
n^5	0		1		32		243		1024		3125		7776
		1		31		211		781		2101		4651	
			30		180		570		1320		2550		
				150		390		750		1230			
					240		360		480				
						120		120					
							0						

4. X: m Unlabeled Balls, Y: n Labeled Boxes, Unrestricted Placement

   (a) Interpretation

   Counts the number of ways of writing m as a sum of n nonnegative integers, where different orderings are counted as different. Let's call these *weak compositions of m into n parts.*

   We made a table and saw the appearance of Pascal's triangle.

   (b) Closed Formula

   Let's use the notation $g(m,n)$ for the number of ways to do this. Think of making a line of $m+n-1$ white balls and then choosing $n-1$ of them to color black as "spacers." So the formula is $g(m,n) = \binom{m+n-1}{n-1}$. This is an example of a bijective proof.

   It also equals $\binom{m+n-1}{m}$. This is sometimes written $\left(\!\binom{n}{m}\!\right)$.

   (c) Recursive Formula

   One recursive formula is $g(m,n) = g(m-1,n) + g(m,n-1)$, which we verified by substituting the closed formula into this expression.

   Another is $g(m,n) = g(m,n-1)+g(m-1,n-1)+g(m-2,n-1)+\cdots+g(0,n-1)$, which we again verified by substituting the closed formula into this expression, though we had to use the binomial recursion $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$ multiple times after changing the last term from $\binom{n-2}{0}$ to $\binom{n-1}{0}$.

   (d) Related Problems

   Counts the number of multisets of $\{1,\ldots,n\}$ of size m.

   Counts the number of monomials of degree m using n variables.

   We observed that $(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)(1+x_3+x_3^2+\cdots)\cdots(1+x_n+x_n^2+\cdots)$ when formally expanded yields all of the monomials in n variables. So, by replacing each of the x_i with the single variable x, the number $g(m,n) = \left(\!\binom{n}{m}\!\right)$ is the coefficient of x^m in the expansion of $(1+x+x^2+\cdots)^n = (\frac{1}{1-x})^n$. This is an example of a *generating function* for the sequence $g(0,n), g(1,n), g(2,n), \ldots$. Try typing "`series 1/(1-x)^5`" or even "`series 1/(1-x)^n`" into www.wolframalpha.com.

   In like manner, we can think of $(1+x)^n$ as a generating function for the finite sequence $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$.

5. X: m Unlabeled Balls, Y: n Labeled Boxes, Injective Placement

   (a) Interpretation

   Counts the number of subsets of $\{1, \ldots, n\}$ of size m.

   (b) Closed Formula

   We denote this number, called a *binomial coefficient*, as $\binom{n}{m}$ or $C(n, m)$ or ${}_nC_m$. We can prove using the multiplication counting principle and permutations, (or the recursive formula below) that

$$\binom{n}{m} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{m(m-1)(m-2)\cdots 1} = \frac{(n)_m}{m!} = \frac{n!}{m!(n-m)!}.$$

   We noted the relationship to the closed formula of (2).

   (c) Recursive Formula

   $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, $n \geq 0$; $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$, $n \geq 1$, $0 < m < n$. This makes sense when thinking of subsets, leading to a bijective proof. It can also be proved algebraically directly from the closed formula.

   (d) Related Problems

   The numbers $\binom{n}{m}$ are the entries in Pascal's Triangle. This follows from the recursive formula. You can also prove this by counting paths from the apex of Pascal's triangle to any particular entry, which count the number of times the 1 at the apex contributes to that entry. We saw a correspondence between the paths and the ways of choosing directions (left or right) along the path.

   $\binom{n}{m}$ is the coefficient of x^m in $(1+x)^n$.

$$(1+x)^n = \sum_{m=0}^{n} \binom{n}{m} x^m.$$

   These make sense when writing out $(1+x)\cdots(1+x)$ and choosing which terms will contribute to x^m. The recursive formula also makes sense when writing out $(1+x)^{n-1}(1+x)$ and seeing which terms will contribute to x^m.

   $\binom{n}{m}$ is the coefficient of $a^{n-m}b^m$ in $(a+b)^n$.

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m.$$

By substituting $x = 1$ into the above formula, or using that there are 2^n subsets of Y, we have

$$\sum_{m=1}^{n} \binom{n}{m} = 2^n.$$

By substituting $x = -1$ into the above formula, we have

$$\sum_{m=1}^{n} (-1)^m \binom{n}{m} = 0,$$

which shows that there are equal numbers of subsets of even cardinality and of odd cardinality.

6. X: m Unlabeled Balls, Y: n Labeled Boxes, Surjective Placement

   (a) Interpretation

   Counts the number of ways of writing m as a sum of n positive integers, where different orderings are counted as different. These are called *compositions* of m.

   (b) Closed Formula

   We can place one ball in each box first, leaving $m-n$ balls to place with no restriction in n boxes. Or we can take a composition of m into n positive integers and and subtract 1 from each term to get an expression of $m-n$ as a sum of n nonnegative integers. Either way, by (4) there are $\left(\!\binom{n}{m-n}\!\right)$ ways to do this, which equals $\binom{m-1}{m-n}$ or $\binom{m-1}{n-1}$.

   We can also argue directly: Think of making a line of $m+n-1$ white balls and then choosing $n-1$ of them to color black as "spacers." But this time we cannot select two spacers side by side, and we cannot select an end ball to be a spacer. So the first ball cannot be black, and every black ball is followed by a white one. Collapse each of the $n-1$ black-white pairs of balls into a single red ball. Now there is a line of m balls, from which you must select $n-1$ of them, but you are not allowed to choose the first one. The number of ways of doing this is $\binom{m-1}{n-1}$.

   (c) Recursive Formula

   If we call the number of surjective placements $h(m,n)$, then the standard binomial identity implies $h(m,n) = h(m-1,n-1) + h(m-1,n)$. Can you find a combinatorial proof of this?

   (d) Related Problems

   Counts the number of monomials of degree m using n variables in which every variable is raised to a positive power.

   Equals the coefficient of x^m in $(x+x^2+x^3+\cdots)^n$ or $\left(\frac{x}{1-x}\right)^n$.

   The total number of compositions of m equals 2^{m-1}, since

$$\sum_{n=1}^{m} \binom{m-1}{n-1} = 2^{m-1}.$$

   Or use a generating function proof by considering

$$1 + (x+x^2+x^3+\cdots) + (x+x^2+x^3+\cdots)^2 + (x+x^2+x^3+\cdots)^3 + \cdots.$$

The coefficient of x^m in this expression equals the total number of compositions of m. But this expression simplifies to

$$\begin{aligned} 1+\frac{x+x^2+x^3+\cdots}{1-(x+x^2+x^3+\cdots)} &= 1+\frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} \\ &= 1+\frac{x}{1-2x} \\ &= 1+x(1+2x+(2x)^2+(2x)^3+\cdots) \\ &= 1+x+2x^2+2^2x^3+2^3x^4+\cdots. \end{aligned}$$

For a bijective proof, map a composition $m = a_1 + \cdots + a_n$ to a collection of partial sums $\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + \cdots + a_{n-1}\} \subseteq \{1, \ldots, m-1\}$.

How many compositions of m are there using only 1's, 2's, 4's, and 8's? We can solve this problem by looking for the coefficient of x^m in the series expansion of

$$1+(x+x^2+x^4+x^8)+(x+x^2+x^4+x^8)^2+(x+x^2+x^4+x^8)^3+\cdots = \frac{1}{1-(x+x^2+x^4+x^8)}.$$

An algebraic calculator like wolframalpha can help us here.

How many compositions of m are there using only 1's and 2's? We saw that the answer was the mth Fibonacci number, because the compositions of m can be obtained from the compositions of $m-2$ by appending "+2" and from the compositions of $m-1$ by appending "+1". The number of such compositions is the coefficient of x^m in the series expansion of

$$1+(x+x^2)+(x+x^2)^2+(x+x^2)^3+\cdots = \frac{1}{1-x^-x^2}.$$

We computed the first few terms of this using wolframalpha, and saw the appearance of the Fibonacci numbers as coefficients.

7. X: m Labeled Balls, Y: n Unlabeled Boxes, Unrestricted Placement

   (a) Interpretation

   This counts the number of *partitions* of X into at most n *blocks*.

   (b) Closed Formula

   By (9) this equals $S(m, 1) + \cdots + S(m, n)$.

   (c) Recursive Formula

   (d) Related Problems

   This counts the number of equivalence relations on X with at most n equivalence classes; hence the total number of equivalence relations if $n \geq m$.

8. X: m Labeled Balls, Y: n Unlabeled Boxes, Injective Placement

   (a) Interpretation

   Either you can do it or you cannot!

   (b) Closed Formula

   1 if $m \leq n$; 0 if $m > n$.

   (c) Recursive Formula

   (d) Related Problems

9. X: m Labeled Balls, Y: n Unlabeled Boxes, Surjective Placement

   (a) Interpretation

   This counts the number of partitions of the set X into exactly n nonempty blocks.

   (b) Closed Formula

   This number is known as $S(m,n)$, a *Stirling number of the second kind.* From (3) we get the formula

$$S(m,n) = \frac{\hat{S}(m,n)}{n!} = \frac{1}{n!}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}i^m.$$

   because the order of the boxes (which are distinguishable by their nonempty contents) is no longer relevant.

   (c) Recursive Formula

   Thinking of partitions,

$$S(m,n) = S(m-1,n-1) + nS(m-1,n).$$

   Think about where to insert element m into a partition of $m-1$ elements. What is the base case?

   (d) Related Problems

   This counts the number of equivalence relations on X with exactly n equivalence classes.

10. X: m Unlabeled Balls, Y: n Unlabeled Boxes, Unrestricted Placement

(a) Interpretation

This is the number of ways of writing m as a sum of at most n positive integers, where the order does not matter—the number of *partitions* of the number m into at most n parts. From (12) this equals $p_1(m) + \cdots + p_n(m)$.

(b) Closed Formula

(c) Recursive Formula

(d) Related Problems

If $n \geq m$ we get the total number, $p(m)$, of partitions of m. A generating function for these numbers is

$$\sum_{m \geq 0} p(m) x^m = \prod_{i \geq 1} (1 - x^i)^{-1}.$$

This is explained by thinking of the number m as written as a multiple of 1, plus a multiple of 2, plus a multiple of 3, etc.

11. X: m Unlabeled Balls, Y: n Unlabeled Boxes, Injective Placement

    (a) Interpretation

        Either you can do it, or you cannot.

    (b) Closed Formula

        1 if $m \leq n$; 0 if $m > n$.

    (c) Recursive Formula

    (d) Related Problems

12. X: m Unlabeled Balls, Y: n Unlabeled Boxes, Surjective Placement

    (a) Interpretation

    This is the number of ways of writing m as a sum of exactly n positive integers, where the order does not matter. These are called *partitions* of the number m into n *parts*. This number is denoted $p_n(m)$.

    (b) Closed Formula

    (c) Recursive Formula

    Either the smallest part equals 1 or it does not. If it does, remove that part. If it does not, subtract 1 from all parts. Thus

    $$p_n(m) = p_{n-1}(m-1) + p_n(m-n).$$

    (d) Related Problems

    We used the cubes to make *Ferrers diagrams* of partitions, and saw how "flipping them" shows that the number of partitions of m into n parts equals the number of partitions of m into an unrestricted number of parts with the largest part equaling n.

2 The Pigeonhole Principle

Quite simply, the pigeonhole principle states that if $|X| > |Y|$, then there are no one-to-one functions from X into Y. So if $m > n$ and there are m pigeons and n pigeonholes in which they roost, there must be at least one pigeonhole with more than one pigeon.

This obvious principle can be used to prove some perhaps less-than-obvious facts.

1. Prove that there are at least two humans on earth with the same number of hairs on their heads.

2. There are 100 people at a party. Assume that if person A knows person B, then person B knows person A. Prove that there are at least two people at the party who know the same number of people.

3. In a bureau drawer there are 60 socks, all identical except for their color: 10 pairs are red, 10 are blue, and 10 are green. The socks are all mixed up in the drawer, and the room the bureau is in is totally dark. What is the smallest number of socks you must remove to be sure you have at least one matching pair?

4. Prove that when a fraction a/b is expressed in decimal form, the resulting number will be either a terminating decimal or one that repeats with a period no longer than b.

5. Prove that if five points are placed anywhere on or in a square of side length 1, at least two points will be no farther apart than $\sqrt{2}/2$.

6. Prove that if five points are placed anywhere on or in an equilateral triangle of side length 1, at least two points will be no farther apart than 1/2.

7. Try counting the edges around the faces of a polyhedron. You will find that there are always two faces somewhere bounded by the same number of edges. Why?

8. Prove that no matter how a set S of 10 positive integers smaller than 100 is chosen there will always be two completely different selections from S that have the same sum. For example, in the set 3, 9, 14, 21, 26, 35, 42, 59, 63, 76 there are the selections 14, 63, and 35, 42, both of which add to 77; similarly, the selection 3, 9, 14 adds up to 26, a number that is a member of the set.

9. A physician testing a new medication instructs a test patient to take 48 pills over a 30-day period. The patient is at liberty to distribute the pills however he likes over this period as long as he takes at least one pill a day and finishes all 48 pills by the end of the 30 days. Prove that no matter how the patient decides to arrange things,

there will be some stretch of consecutive days in which the total number of pills taken is 11.

10. Suppose some set of 101 numbers $a_1, a_2, \ldots, a_{101}$ is chosen from the numbers $1, 2, \ldots, 200$. Prove that it is impossible to choose such a set without taking two numbers for which one divides the other evenly; that is, with no remainder.

11. Consider a circle C with a radius of 16 and an annulus, or ring, A, with an outer radius of 3 and an inner radius of 2. Prove that wherever one might sprinkle a set S of 650 points inside C the annulus A can always be placed on the figure so that it covers at least 10 of the points.

12. The next example concerns a marching band whose members are lined up in a rectangular array of m rows and n columns. Viewing the band from the left side, the bandmaster notices that some of the shorter members are hidden in the array. He rectifies this aesthetic flaw by arranging the musicians in each row in nondecreasing order of height from left to right, so that each one is of height greater than or equal to that of the person to his left (from the viewpoint of the bandmaster). When the bandmaster goes around to the front, however, he finds that once again some of the shorter members are concealed. he proceeds to shuffle the musicians within their columns so that they are arranged in nondecreasing order of height from front to back. At this point he hesitates to go back to the left side to see what this adjustment has done to his carefully arranged rows. When he does go, however, he is pleasantly surprised to find that the rows are still arranged in nondecreasing order of height from left to right! Shuffling an array within its columns in this manner does not undo the nondecreasing order in its rows. Can you prove this?

13. Take the numbers from $1, 2, 3, \ldots, n^2 + 1$ and arrange them in a sequence in any order. Prove that when the arrangement is scanned from left to right, it must contain either an increasing subsequence of length (at least) $n + 1$ or a decreasing subsequence of length (at least) $n + 1$. For example, when $n = 3$, the arrangement $6, 5, 9, 3, 7, 1, 2, 8, 4, 10$ includes the decreasing subsequence $6, 5, 3, 1$. (As this example demonstrates, a subsequence need not consist of consecutive elements of the arrangement.)

14. A lattice point is a point in a coordinate plane for which both coordinates are integers. Prove that no matter what five lattice points might be chosen in the plane at least one of the segments that joins two of the chosen points must pass through some lattice point in the plane.

15. Six circles (including their circumferences and interiors) are arranged in the plane so that no one of them contains the center of another. Prove that they cannot have a point in common.

16. Prove that in any row of $mn+1$ distinct real numbers there must be either an increasing subsequence of length (at least) $m+1$ or a decreasing subsequence of length (at least) $n+1$.

These problems were taken from Martin Gardner, The power of the pigeonhole, *The Last Recreations*, chapter 11. The answers to the problems can be found there.

3 Guessing Formulas

> It should be remarked that although the principle of mathematical induction suffices to *prove* the formula... once this formula has been written down, the proof gives no indication of how this formula was arrived at in the first place; why precisely the expression $[n(n+1)/2]^2$ should be guessed as an expression for the sum of the first n cubes, rather than $[n(n+1)/3]^2$ or $(19n^2-41n+24)/2$ or any of the infinitely many expressions of a similar type that could have been considered. The fact that the proof of a theorem consists in the application of certain simple rules of logic does not dispose of the creative element in mathematics, which lies in the choice of the possibilities to be examined. The question of the origin of the *hypothesis*... belongs to a domain in which no very general rules can be given; experiment, analogy, and constructive intuition play their part here. But once the correct hypothesis is formulated, the principle of mathematical induction is often sufficient to provide the proof. Inasmuch as such a proof does not give a clue to the act of discovery, it might more fittingly be called a *verification*.
> —Courant and Robbins, *What is Mathematics*, Section I.2.4.

In this section we discuss some ways to guess formulas for sequences. (P.S. Courant and Robbins is an excellent book to add to your personal library.)

3.1 Finite Differences

Suppose you are asked to find a function $f(x)$ such that

$$\begin{array}{lcr} f(0) & = & -7 \\ f'(0) & = & 5 \\ f''(0) & = & -6 \\ f'''(0) & = & 12 \\ f^{(4)}(0) & = & 0 \\ f^{(5)}(0) & = & 0 \\ f^{(6)}(0) & = & 0 \\ & \vdots & \end{array}$$

We might guess that the function is a polynomial of degree 3. How can we determine the coefficients?

$$\begin{array}{lcl} f(x) & = & c_0 + c_1x + c_2x^2 + c_3x^3 \\ f'(x) & = & c_1 + 2c_2x + 3c_3x^2 \\ f''(x) & = & 2c_2 + 6c_3x \\ f'''(x) & = & 6c_3 \end{array} \qquad \begin{array}{lcl} f(0) & = & c_0 \\ f'(0) & = & c_1 \\ f''(0) & = & 2c_2 \\ f'''(0) & = & 6c_3 \end{array} \qquad \begin{array}{lcl} c_0 & = & f(0) \\ c_1 & = & f'(0) \\ c_2 & = & f''(0)/2 \\ c_3 & = & f'''(0)/6 \end{array}$$

In our example, $c_0 = -7$, $c_1 = 5$, $c_2 = -3$, and $c_4 = 2$ so $f(x) = -7 + 5x - 3x^2 + 2x^3$.

In general, if you think $f(x)$ is a polynomial of degree m, $f(x) = c_0+c_1x+c_2x^2+\cdots+c_mx^m$, then

$$\begin{array}{lcl} c_0 & = & f(0)/0! \\ c_1 & = & f'(0)/1! \\ c_2 & = & f''(0)/2! \\ & \vdots & \\ c_m & = & f^{(m)}(0)/m! \end{array}$$

so

$$f(x) = f(0)\frac{x^0}{0!} + f'(0)\frac{x^1}{1!} + f''(0)\frac{x^2}{2!} + \cdots + f^{(m)}(0)\frac{x^m}{m!}.$$

What is the number of different triangles you can form on a flat surface using three toothpicks for the three sides of each triangle, given an unlimited supply of toothpicks of n

different colors?

number of colors	number of triangles
0	0
1	1
2	4
3	11
4	24
5	45
$\vdots$	$\vdots$

We want a formula $f(n)$ so that

$$f(0), f(1), f(2), f(3), f(4), f(5), \ldots = 0, 1, 4, 11, 24, 45, \ldots.$$

Look for a pattern by subtracting these numbers from each other, making a difference table:

$$\begin{array}{ccccccccccccc}
0 & & 1 & & 4 & & 11 & & 24 & & 45 & & \cdots \\
& 1 & & 3 & & 7 & & 13 & & 21 & & \cdots & \\
& & 2 & & 4 & & 6 & & 8 & & \cdots & & \\
& & & 2 & & 2 & & 2 & & \cdots & & & \\
& & & & 0 & & 0 & & \cdots & & & &
\end{array}$$

Consider some known formulas:

$$f(n) = n^2 : \quad \begin{array}{ccccccccccccc}
0 & & 1 & & 4 & & 9 & & 16 & & 25 & & \cdots \\
& 1 & & 3 & & 5 & & 7 & & 9 & & \cdots & \\
& & 2 & & 2 & & 2 & & 2 & & \cdots & & \\
& & & 0 & & 0 & & 0 & & \cdots & & &
\end{array}$$

$$f(n) = n^3 - n : \quad \begin{array}{ccccccccccccc}
0 & & 0 & & 6 & & 24 & & 60 & & 120 & & \cdots \\
& 0 & & 6 & & 18 & & 36 & & 60 & & \cdots & \\
& & 6 & & 12 & & 18 & & 24 & & \cdots & & \\
& & & 6 & & 6 & & 6 & & \cdots & & & \\
& & & & 0 & & 0 & & \cdots & & & &
\end{array}$$

This suggests that for our problem we seek a formula of degree 3.

Let's call the numbers in the first row

$$f(0), f(1), f(2), f(3), f(4), \ldots$$

and the numbers in the second row

$$f'(0), f'(1), f'(2), f'(3), f'(4), \ldots$$

and the numbers in the third row

$$f''(0), f''(1), f''(2), f''(3), f''(4), \ldots$$

etc. These aren't equal to derivatives in the sense of differential calculus, but there seems to be a strong analogy.

$$f'(n) = \frac{f(n+1) - f(n)}{1} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In differential calculus, we exploited;

$$\begin{array}{cc} \text{function} & \text{derivative} \\ x^0 & 0 \\ x^1 & 1x^0 \\ x^2 & 2x^1 \\ x^3 & 3x^2 \\ \vdots & \vdots \\ x^k & kx^{k-1} \end{array}$$

What is the analog for differences? Define

$$n^{\underline{k}} = \underbrace{n(n-1)(n-2)\cdots(n-k+1)}_{k \text{ terms}}.$$

This is the falling factorial that we say before.

$$\begin{array}{cccc} & \text{function} & \text{“derivative”} & \\ 1 & n^{\underline{0}} & 0 & 0 \\ n & n^{\underline{1}} & 1n^{\underline{0}} & 1 \\ n(n-1) & n^{\underline{2}} & 2n^{\underline{1}} & 2n \\ n(n-1)(n-2) & n^{\underline{3}} & 3n^{\underline{2}} & 3n(n-1) \\ \vdots & \vdots & \vdots & \vdots \\ n(n-1)\cdots(n-k+1) & n^{\underline{k}} & kn^{\underline{k-1}} & kn(n-1)\cdots(n-k+2) \end{array}$$

Verification:

$$\begin{aligned} (n^{\underline{k}})' &= (n+1)^{\underline{k}} - n^{\underline{k}} \\ &= (n+1)(n)(n-1)\cdots(n-k+2) - n(n-1)\cdots(n-k+2)(n-k+1) \\ &= n(n-1)\cdots(n-k+2)\underbrace{((n+1)-(n-k+1))}_{k} \\ &= kn^{\underline{k-1}}. \end{aligned}$$

Now we can guess formulas:

$$
\begin{aligned}
f(n) &= c_0 + c_1 n + c_2 n^{\underline{2}} + c_3 n^{\underline{3}} & f(0) &= c_0 & c_0 &= f(0) \\
f'(n) &= c_1 + 2c_2 n + 3c_3 n^{\underline{2}} & f'(0) &= c_1 & c_1 &= f'(0) \\
f''(n) &= 2c_2 + 6c_3 n & f''(0) &= 2c_2 & c_2 &= f''(0)/2 \\
f'''(n) &= 6c_3 & f'''(0) &= 6c_3 & c_3 &= f'''(0)/6
\end{aligned}
$$

In our example,

$$
\begin{aligned}
f(0) &= 0 & c_0 &= 0 \\
f'(0) &= 1 & c_1 &= 1 \\
f''(0) &= 2 & c_2 &= 1 \\
f'''(0) &= 2 & c_3 &= 1/3
\end{aligned}
$$

$$
\begin{aligned}
f(n) &= 0 + 1n^{\underline{1}} + 1n^{\underline{2}} + \frac{1}{3}n^{\underline{3}} \\
&= n + n(n-1) + \frac{n(n-1)(n-2)}{3} \\
&= \frac{n^3 + 2n}{3}.
\end{aligned}
$$

Now we have a formula that we can try to prove, e.g., by induction, or directly.

In general, fetch $f(0)$, $f'(0)$, $f''(0)$, $\ldots$, $f^{(k)}(0)$ as the first entries of the rows of the difference table (assuming we have reached a row of zeroes). Then

$$
\begin{aligned}
c_0 &= f(0)/0! \\
c_1 &= f'(0)/1! \\
c_2 &= f''(0)/2! \\
&\vdots \\
c_k &= f^{(k)}(0)/k!
\end{aligned}
$$

$$
\begin{aligned}
f(n) &= c_0 + c_1 n + c_2 n^{\underline{2}} + \cdots + c_k n^{\underline{k}} \\
&= f(0)\frac{n^{\underline{0}}}{0!} + f'(0)\frac{n^{\underline{1}}}{1!} + f''(0)\frac{n^{\underline{2}}}{2!} + \cdots + f^{(k)}(0)\frac{n^{\underline{k}}}{k!}.
\end{aligned}
$$

Note that

$$
\frac{n^{\underline{j}}}{j!} = \frac{n(n-1)\cdots(n-j+1)}{j(j-1)\cdots 3\cdot 2\cdot 1} = \binom{n}{j}
$$

so

$$
f(n) = f(0)\binom{n}{0} + f'(0)\binom{n}{1} + f''(0)\binom{n}{2} + \cdots + f^{(k)}(0)\binom{n}{k},
$$

(taking $\binom{n}{k} = 0$ if $n < j$).

In our example,

$$f(n) = 0\binom{n}{0} + 1\binom{n}{1} + 2\binom{n}{2} + 2\binom{n}{3}.$$

Here is another way of doing the same thing—via “antiderivatives.”

$$f''''(n) = 0.$$

$$f'''(n) = K.$$

$$f'''(0) = 2 \Longrightarrow 2 = K \Longrightarrow f'''(n) = 2 = 2n^{\underline{0}}.$$

$$f''(n) = 2n^{\underline{1}} + L.$$

$$f''(0) = 2 \Longrightarrow 2 = L \Longrightarrow f''(n) = 2n^{\underline{1}} + 2n^{\underline{0}}.$$

$$f'(n) = n^{\underline{2}} + 2n^{\underline{1}} + M.$$

$$f'(0) = 1 \Longrightarrow 1 = M \Longrightarrow f'(n) = n^{\underline{2}} + 2n^{\underline{1}} + n^{\underline{0}}.$$

$$f(n) = \frac{1}{3}m^{\underline{3}} + n^{\underline{2}} + n^{\underline{1}} + N.$$

$$f(0) = 0 \Longrightarrow 0 = N.$$

$$f(n) = \frac{1}{3}n^{\underline{3}} + n^{\underline{2}} + n^{\underline{1}}.$$

and you can verify that this is equal to $2\binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$, which is the formula we had found before.

As another example, let’s obtain a formula for $f(n) = 0^2 + 1^2 + 2^2 + \cdots + n^2$. Make a difference table:

0		1		2		3		4		5
0		1		5		14		30		55
	1		4		9		16		25	
		3		5		7		9		
			2		2		2			
				0		0				

So

$$\begin{aligned} f(n) &= 0\binom{n}{0} + 1\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3} \\ &= n + 3\frac{n(n-1)}{2} + 2\frac{n(n-1)(n-2)}{6} \\ &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \\ &= \frac{1}{6}n(n+1)(2n+1). \end{aligned}$$

Finally, recall in Case (3) of the Twelve-Fold Way we saw the formula $n^m = \sum_{i=0}^{n} \hat{S}(m,i)\binom{n}{i}$. Our analysis of finite differences now shows why we can expect to see the numbers $\hat{S}(m,i)$ in the first diagonal of the difference table for the function n^m.

3.2 Exponentials

What about the following sequence?

$$\begin{array}{ccccccccccccc}
2 & & 7 & & 30 & & 125 & & 508 & & 2043 & & \cdots \\
& 5 & & 23 & & 95 & & 383 & & 1535 & & \cdots & \\
& & 18 & & 72 & & 288 & & 1152 & & \cdots & &
\end{array}$$

If we take quotients of entries in the last row instead of differences, we see that every quotient is 4. So $f''(n)$ is a geometric, not an arithmetic sequence (as in our earlier examples).

$$f''(n) = 18 \cdot 4^n.$$

What is the "antiderivative" of 4^n? Maybe we can figure this out if we can determine the "derivative" of 4^n.

$$4^{n+1} - 4^n = 4^n(4-1) = 3 \cdot 4^n.$$

So the "antiderivative" of 4^n is $\frac{1}{3}4^n$.

Now we can find a formula for the sequence.

$$f''(n) = 18 \cdot 4^n$$

$$f'(n) = 6 \cdot 4^n + K$$

$$f'(0) = 5 \Longrightarrow 5 = 6 + K \Longrightarrow K = -1 \Longrightarrow f'(n) = 6 \cdot 4^n - [n]^0$$

$$f(n) = 2 \cdot 4^n - [n]^1 + L$$

$$f(0) = 2 \Longrightarrow 2 = 2 + L \Longrightarrow L = 0 \Longrightarrow f(n) = 2 \cdot 4^n - n$$

This method works in general if $f^{(k)}(n)$ is geometric.

3.3 Using Series

Playing around with series can give more techniques. Remember how to figure out the formula for geometric series like

$$h(x) = 1 + 2x + (2x)^2 + (2x)^3 + \cdots$$

$$\begin{aligned} h(x) &= 1 + 2x + (2x)^2 + (2x)^3 + \cdots \\ -2xh(x) &= \quad -2x - (2x)^2 - (2x)^3 - \cdots \end{aligned}$$

$$(1-2x)h(x) = 1$$

$$h(x) = \frac{1}{1-2x} = (1-2x)^{-1}$$

We can keep this in mind as we tackle a "Fibonacci"-like sequence:

$$\begin{array}{ccccccccccccc} 1 & & 1 & & 3 & & 5 & & 11 & & 21 & & 43 \ \cdots \\ & 0 & & 2 & & 2 & & 6 & & 10 & & 22 \ \cdots & \\ & & 2 & & 0 & & 4 & & 4 & & 12 \ \cdots & & \\ & & & -2 & & 4 & & 0 & & 8 \ \cdots & & & \end{array}$$

We never seem to get a row of zeroes. The second row "looks like" twice the first row; i.e., $f(n+1) = f(n) + 2f(n-1)$ for $n \geq 1$. (The ordinary Fibonacci sequence satisfies $f(n+1) = f(n) + f(n-1)$.)

Let define a power series using $f(n)$ as the coefficient of x^n:

$$g(x) = 1 + x + 3x^2 + 5x^3 + 11x^4 + \cdots$$

The relationship $f(n+1) - f(n) - 2f(n-1) = 0$ suggests:

$$\begin{aligned} g(x) &= 1 + x + 3x^2 + 5x^3 + 11x^4 + \cdots \\ -xg(x) &= \quad -x - x^2 - 3x^3 - 5x^4 - \cdots \\ -2x^2g(x) &= \quad -2x^2 - 2x^3 - 6x^4 - \cdots \end{aligned}$$

$$(1 - x - 2x^2)g(x) = 1$$

$$g(x) = \frac{1}{-2x^2 - x + 1}$$

Remember, $f(n)$ is the coefficient of x^n. Let's try to find it.

$$g(x) = \frac{1}{-2x^2 - x + 1} = \frac{1}{(1-2x)(1+x)} = \frac{2/3}{1-2x} + \frac{1/3}{1+x}$$

We did the last step using the method of partial fractions. Continuing,

$$= (2/3)(1-2x)^{-1} + (1/3)(1+x)^{-1}$$

$$\begin{array}{l}(2/3)(1+2x+(2x)^2+(2x)^3+\cdots \\ +(1/3)(1-x+x^2-x^3+\cdots\end{array}$$

since we have geometric series.

So the coefficient of x^n is

$$(2/3)2^n + (1/3)(-1)^n$$

and this is our guess for the formula for $f(n)$.

We then also derived a closed formula for the ordinary Fibonacci sequence this way.

3.4 Using Matrices

Let's look at the previous sequence $1, 1, 3, 5, 11, 21, 43, \ldots$ another way. Make vectors out of pairs of adjacent elements,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 5 \\ 11 \end{bmatrix} \quad \cdots$$

and find a matrix that transforms each vector into the next, using the "Fibonacci" nature of the sequence.

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

In general,

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

and calculate A^n by diagonalizing A

$$\begin{array}{cc} \text{eigenvalues} & \text{eigenvectors} \\ 2 & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ -1 & \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array}$$

$$A \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \\ &= SDS^{-1} \end{aligned}$$

$$
\begin{aligned}
A^n &= (SDS^{-1})(SDS^{-1})\cdots(SDS^{-1}) \\
&= SD^nS^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}
\end{aligned}
$$

So

$$
\begin{aligned}
A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} (2/3)2^n + (1/3)(-1)^n \\ (4/3)2^n - (1/3)(-1)^n \end{bmatrix} \\
&= \begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix}
\end{aligned}
$$

Therefore

$$f(n) = \frac{2}{3}2^n + \frac{1}{3}(-1)^n.$$

Exercise: Try to derive a formula for the ordinary Fibonacci sequence this way.

4 Principle of Inclusion/Exclusion

This relates to Section 3.7 of the text.

Summary of class activities:

1. Considered the problem of finding the number of integers remaining in the set $\{1, \ldots, 210\}$ once all numbers that are multiples of 2, 3, and/or 5 are removed.

2. Used this to motivate the Principle of Inclusion/Exclusion (PIE), a formula to compute $|A_1 \cup \cdots \cup A_n|$ for a collection of sets. We saw three ways to formulate this principle.

3. Illustrated the first proof (by induction) given in the book with the "sufficiently complicated example" of $|A_1 \cup A_2 \cup A_3 \cup A_4|$.

4. Went through the second proof given in the book, in which the contribution of an arbitrary element $x \in A_1 \cup \cdots \cup A_n$ to the sum is computed and shown to be 1.

5 Derangements

This relates to Section 3.8 of the text.

Summary of class activities:

1. Defined derangements and computed the number $D(n)$ of derangements for small values of n.

2. Detected (!), and then proved, the recursive formula $D(n) = (n-1)(D(n-1)+D(n-2))$. Sketch: Think of a derangement as a placement of balls $1, \ldots, n$ in labeled boxes $1, \ldots, n$ for which no ball is placed in a box with the same label. You can make a derangement of size n in two ways:

   (a) Choose a derangement of $1, \ldots, n-1$ (there are $D(n-1)$ ways to do this), place ball n in box n, and then swap ball n with one of the other balls (there are $n-1$ choices).

   (b) Choose a ball $k \in \{1, \ldots, n-1\}$ (there are $n-1$ choices), place ball k in box k, choose a derangement of balls $\{1, \ldots, n-1\} \setminus \{k\}$ in the boxes $\{1, \ldots, n-1\} \setminus \{k\}$ (there are $D(n-2)$ ways to do this), and then finally swap ball n with ball k.

   Each derangement of size n is obtainable by exactly one of the above two procedures, in exactly one way.

3. Discussed the closed formula for derangements obtained from the Principle of Inclusion/Exclusion.

4. Showed that the probability that a random permutation of n elements is a derangement is approximately $\frac{1}{e}$.

6 Graphs

Summary of class activities:

1. Basic definitions, a bit of graph isomorphism, subgraphs, connectivity, components, the adjacency matrix, walks, paths, and cycles (Sections 4.1 and 4.2 of the text).

2. Making "random graphs" by exchanging "business cards."

3. Using a queue to carry out breadth-first search on a graph to determine the vertices in a component of a graph.

4. Determining the distance between pairs of vertices when the edges have each been assigned a nonnegative cost, using a modification of matrix multiplication — see Homework #4.

5. Eulerian graphs — Section 4.4 of the text. Alternative proof of Theorem 4.4.1. Assume that G is a connected graph with each vertex of even degree. Begin at any vertex v and start walking, never reusing an edge. Eventually you will return to v. If all edges have been used, then you have found an Eulerian closed walk W. Otherwise, upon deletion of this walk, the remaining components each have Eulerian closed walks by induction on the number of edges. Show that each such component shares at least one vertex with W. "Splice" the Eulerian closed walks from each component into W. This can be turned into an algorithm using a stack.

6. Traversing mazes using pebbles, from Martin Gardner's book *The Second Scientific American Book of Mathematical Puzzles and Diversions*, Chapter 10.

7 Trees and Spanning Trees

Summary of class activities:

1. Definitions of trees and spanning trees. Every tree has at least two vertices of degree 1. See Section 5.1 of the text.

2. The minimum spanning tree problem and Kruskal's "Greedy" algorithm (though we did not prove the correctness of this algorithm) — see Section 5.4 of the text.

3. The number of spanning trees in the complete graph K_n is n^{n-2}. Proof using Prüfer codes — see Section 8.4.

4. Matrix Tree Theorem for the number of spanning trees in an arbitrary graph (which we did not prove) — see Section 8.5 of the text.

5. Deletion and contraction formula for the number of spanning trees: $t(G) = t(G \setminus e) + t(G \backslash e)$.

6. A brief introduction to the relationship between the set of triangulations of a convex polygon, rooted binary trees, ways to insert parentheses into a string of symbols, and certain paths in a square grid (Dyck paths). Some of this material is in Section 12.4 of the text.

7. An application of triangulations and trees to prove the Art Gallery Theorem — the number of guards needed to guard an art gallery in the form of a simple polygon with n vertices is at most $n/3$.

8 Matchings

Summary of class activities:

1. An algorithm, by growing alternating trees, to find a maximum cardinality matching in a bipartite graph. König's Theorem: In a bipartite graph, the size of a maximum cardinality matching equals the size of a minimum cardinality set of vertices that collectively are incident to every edge of the graph (a "covering of the edges by vertices").

2. An algorithm to find a stable matching in a bipartite graph. For example, see the website http://www.ams.org/featurecolumn/archive/marriage.html.

9 Chromatic Polynomials

Summary of class activities:

1. Let $\chi_G(k)$ be the number of ways of coloring the vertices of a graph G using at most k colors, so that no two vertices of the same color are joined by an edge.

2. Proved the formula $\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G\setminus e}(k)$. Used this to prove that if $|V| = n$, then $\chi_G(k)$ is a polynomial in k of degree n, with leading term k^n.

10 Combinatorial Games

Summary of class activities:

1. Brief introduction to finite, two-person combinatorial games.

2. If there can be no tie, then either the first player or else the second player has a winning strategy. A position is "good" if it one that you want to attain; otherwise the position is "bad." The final winning positions are good and the final losing positions are bad. An intermediate position is good if every immediately subsequent position your opponent can attain is bad; otherwise it is good. With this notion it is possible to analyze some simple games.

3. A game is *impartial* if both players have the same options from any given position. Nim values (or Sprague-Grundy values) can be assigned to positions. These values are nonnegative integers. Final winning positions are labeled 0. Intermediate positions are labeled with the smallest integer absent from the values of immediately subsequent positions (the mex or minimum-excluded rule).

4. You can define the addition $G_1 + G_2$ of two games with the rule that when it is your turn to make a move in $G_1 + G_2$ you must choose exactly one of the games and make a valid move in that game. A useful theorem for the addition of impartial games is that the nim value of the sum of two (or more) games is the nim-sum of the nim values of these games. The nim-sum of two nonnegative integers is obtained by writing the integers in base two, adding them without carrying, and then converting them back to base ten.

5. In this way, for example, you can win at the game of Nim, in which there are several piles of chips, and when it is your turn you must take any positive number of chips from any one pile of your choice. The person to take the last chip of all is the winner. The value of any position is found by taking the nim-sum of the numbers of chips in the various piles. The winning strategy is to always move to a position of nim-value zero.