MA515 EXAM #2
Due Monday, November 1, 4:30 pm
Give it to me personally, or else slide it under my office door

You may consult the course notes and the course text, and you may ask me questions, but you may not use any other source of information, either human or nonhuman.

1. Recall that a subset of $\mathbf{R}^n$ is defined to be *convex* if it is closed under all finite convex combinations. Recall also that if $X \subseteq \mathbf{R}^n$, then $\text{conv}(X)$ is defined to be the set of all convex combinations of points in X.

   (a) Prove that $\text{conv}(X)$ is convex.

   (b) Prove that the intersection of any collection of convex sets is convex. (Note that we permit an infinite, even an uncountably infinite, number of convex sets in the collection.)

   (c) Prove that $\text{conv}(X)$ equals the intersection of all convex sets containing X.

2. Assume that $x^1, \ldots, x^N \in \mathbf{R}^n$, $x \in \mathbf{R}^n$, and x is a convex combination of $x^1, \ldots, x^N$.

   (a) Prove that there exists a subset S of $\{x^1, \ldots, x^N\}$ of size at most $n+1$ such that x is a convex combination of the points in S. Suggestion: First express the fact that x is a convex combination of $x^1, \ldots, x^N$ as a system in the form:

   $$\begin{aligned} Aw &= b \\ w &\geq O \end{aligned}$$

   (b) Illustrate this theorem by an example in the case $n = 2$.

3. Consider the following six points in $\mathbf{R}^3$: $x^1 = (1, 0, 0)$, $x^2 = (-1, 0, 0)$, $x^3 = (0, 1, 0)$, $x^4 = (0, -1, 0)$, $x^5 = (0, 0, 1)$, and $x^6 = (0, 0, -1)$. Let M be the matroid defined by $E(M) = \{1, 2, 3, 4, 5, 6\}$, and a subset $X \subseteq E(M)$ is an independent for the matroid iff $\{x^i : i \in X\}$ is *affinely independent*.

   (a) What is the rank of M? The rank of M^*?

   (b) List the circuits of M.

   (c) List the circuits of M^*.

4. Do the Exercises on page 58 and on page 69 in Jon Lee's book.

5. Consider the linear matroid represented by the columns of the following matrix over the field of real numbers:

$$\begin{bmatrix} -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Write down a complete set of inequalities to describe the associated matroid polytope. I do not require that you discard redundant constraints (those implied by the other constraints), but you may do so if you wish.