MA515 Homework #6
Due Wednesday, October 13

1. Exercise 10.11 from my notes.

2. Problem (Unique-circuit property), page 53 of the Jon Lee book.

3. Exercise (Linear over GF(2) does not imply graphic), page 54 of the Jon Lee book.

4. Exercise (Nonrepresentable matroids), page 56 of the Jon Lee book.

5. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and no loops. A *walk* in G is a sequence of vertices and edges, $v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k$ such that e_i has endpoints v_{i-1} and v_i, $i = 1, \ldots, k$. Two vertices v and w are in the same *component* of G if there is walk from v to w. A graph with exactly one component is said to be *connected*. A *cycle* in G is a walk with at least two edges, no repeated edges, and no repeated vertices except that $v_k = v_0$. A graph is *acylic* or a *forest* if it does not contain any cycles. A graph is a *tree* if it is a connected forest. Orient each edge of G arbitrarily, so that each edge has one endpoint designated as its *tail* and the other as its *head*. Define the matrix B, with rows indexed by the vertices of G and the columns indexed by the edges of G, by $B_{ve} = -1$ if v is the tail of e, 1 if v is the head of e, and 0 otherwise. This matrix B is to be regarded as a matrix over the real numbers. Solve the following problems, noting the useful linear algebra facts that are listed at the end of the assignment. Most of what you need for (c)–(f) can be derived from (a)–(b).

   (a) Prove that a subset of columns of B is linearly dependent iff the corresponding set of edges of G contains a cycle.

   (b) Prove that a subset of rows of B is linearly dependent iff the corresponding set of vertices of G contains all of the vertices of at least one component of G.

   (c) Prove that G is connected iff $\dim\{y : y^T B = O^T\} = 1$ iff $\text{rank}(B) = |V(G)| - 1$.

   (d) Prove that G is acylic iff $\dim\{x : Bx = O\} = 0$ iff $\text{rank}(B) = |E(G)|$.

   (e) Prove that G contains a unique cycle iff $\dim \{x : Bx = O\} = 1$ iff $\text{rank}(B) = |E(G)| - 1$.

   (f) Prove that the following statements are equivalent:

      i. G is a tree.

      ii. G is connected and $|E(G)| = |V(G)| - 1$.

      iii. G is connected, but deleting any edge of G disconnects G into precisely two components.

iv. G is acyclic and $|E(G)| = |V(G)| - 1$.

v. G is acyclic, but adding a new edge between any pair of vertices in $V(G)$ creates a unique cycle.

Here are some useful facts from linear algebra, for a matrix M with m rows and n columns:

- The *column rank* of M is the maximum size of a linearly independent subset of columns. The *row rank* of M is the maximum size of a linearly independent subset of rows. It is a theorem that the column rank of M always equals the row rank of M, so we call this simply the *rank* of M.

- The *nullspace* of M is $\{x : Mx = O\}$. The *leftnullspace* of M is $\{y : y^T M = O^T\}$. It is a theorem that $\text{rank}(M)+\dim(\text{nullspace}(M)) = n$ and $\text{rank}(M)+\dim(\text{leftnullspace}(M)) = m$.