# Games for Math Circle

Carl W. Lee
Spring 2017

Note: Some of the text below comes from Martin Gardner's articles in *Scientific American* and some from *Mathematical Circles* by Fomin, Genkin, and Itenberg.

## 1 Fifteen

This is a two player game. Take a set of nine cards, numbered one (ace) through nine. They are spread out on the table, face up. Players alternately select a card to add their hands, which they keep face up in front of them. The goal is to achieve a subset of three cards in your hand so that the values of these three cards sum to exactly fifteen. (The ace counts as 1.)

## 2 Ones and Twos

Ten 1's and ten 2's are written on a blackboard. In one turn, a player may erase (or cross out) any two numbers. If the two numbers erased are identical, they are replaced with a single 2. If they are different, they are replaced with a 1. The first player wins if a 1 is left at the end, and the second player wins if a 2 is left.

## 3 100

This is a two player game. Begin with the number zero. Players alternately add a positive whole number from 1 to 6, inclusive, to the current running sum. The first player to exactly achieve the number 100 wins.

## 4 Two Pile Nim

This two person game is played with two piles of 10 coins each. On your move you select one of the piles and take away a positive number of coins from that pile. The winner is the player who takes the very last coin from the table.

# 5 Simultaneous Chess

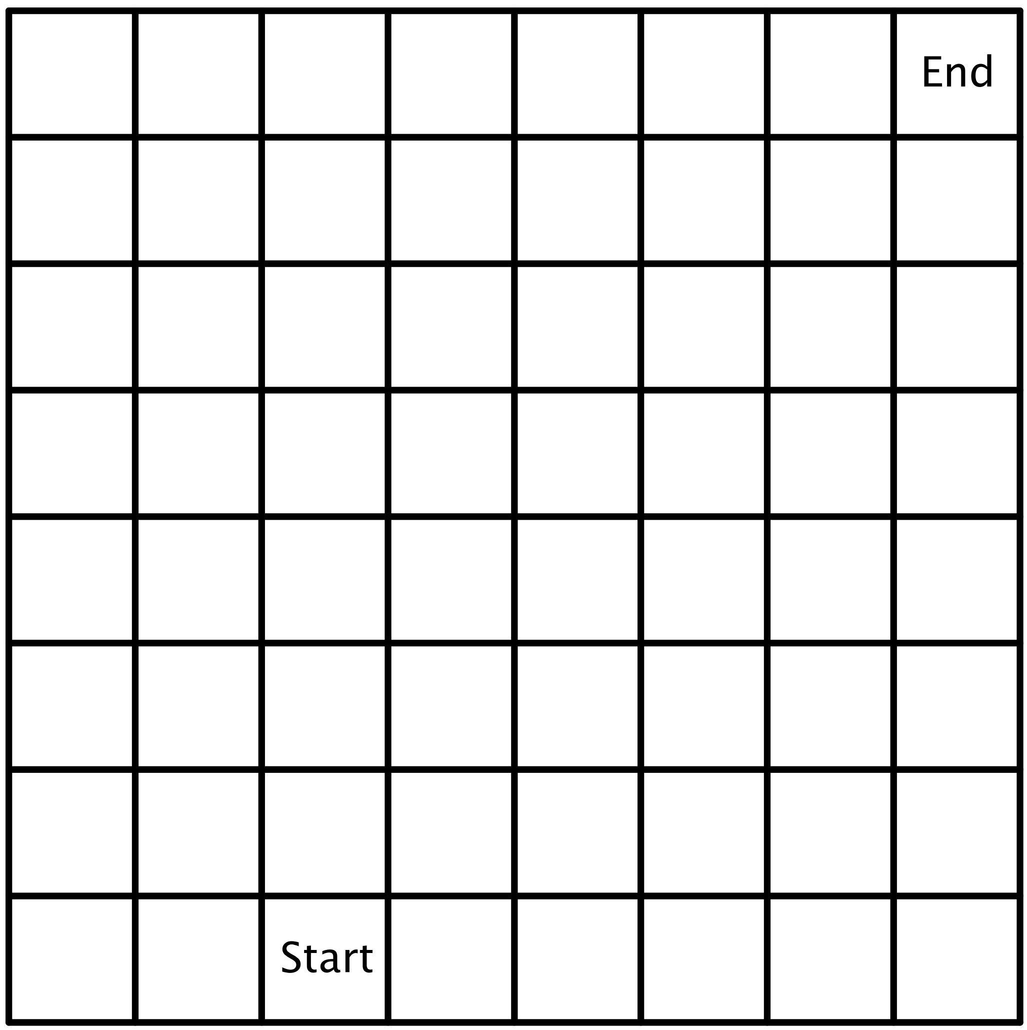
This is really more of a puzzle. How can you play two games of chess simultaneously against two different opponents, and guarantee either winning at least one or else tying both?

# 6 Two Pile Nim II

This two person game is played with two piles of 10 coins each. On your move you may either take exactly one coin from one of the piles, or you may take exactly one coin from each of the piles. The winner is the player who takes the very last coin from the table.

# 7 Queen

Place a queen on the bottom row, third cell from the left (c1) of an $8 \times 8$ chessboard. Players alternate by moving the queen a positive amount right, upward, or diagonally right and upwards. To win you must move the queen to the upper right cell (h8).


End
Start

## 8 Three Pile Nim

This time the game begins with three piles of coins, of sizes 3, 5, and 7. Two players alternately select a pile and remove a positive number of coins from the chosen pile. The player to remove the very last coin wins.

## 9 Kayles

Begin with a row of 10 touching coins. Think of these as bowling pins. Players alternately remove either a single coin, or two touching coins. The first player to take the very last coin wins.

## 10 Hex

Hex is played on a diamond-shaped board made up of hexagons (see Figure 1). The number of hexagons may vary, but the board usually has 11 on each edge. Two opposite sides of the diamond are labeled "black"; the other two sides are "white." The hexagons at the corners of the diamond belong to either side. One player has a supply of black pieces; the other, a supply of white pieces. The player alternately place one of their pieces on any one of the hexagons, provided the cell is not already occupied by another piece. The objective of "black" is to complete an unbroken chain of black pieces between the two sides labeled "black." "White" tries to complete a similar chain of white pieces between the sides labeled "white." See Figure 2 for an example of a path. There will almost certainly be pieces not on the winning path, and there is no obligation to construct the path in any particular order. All that is necessary to win is to have a path joining your two sides of the board somewhere among all the pieces you have played.

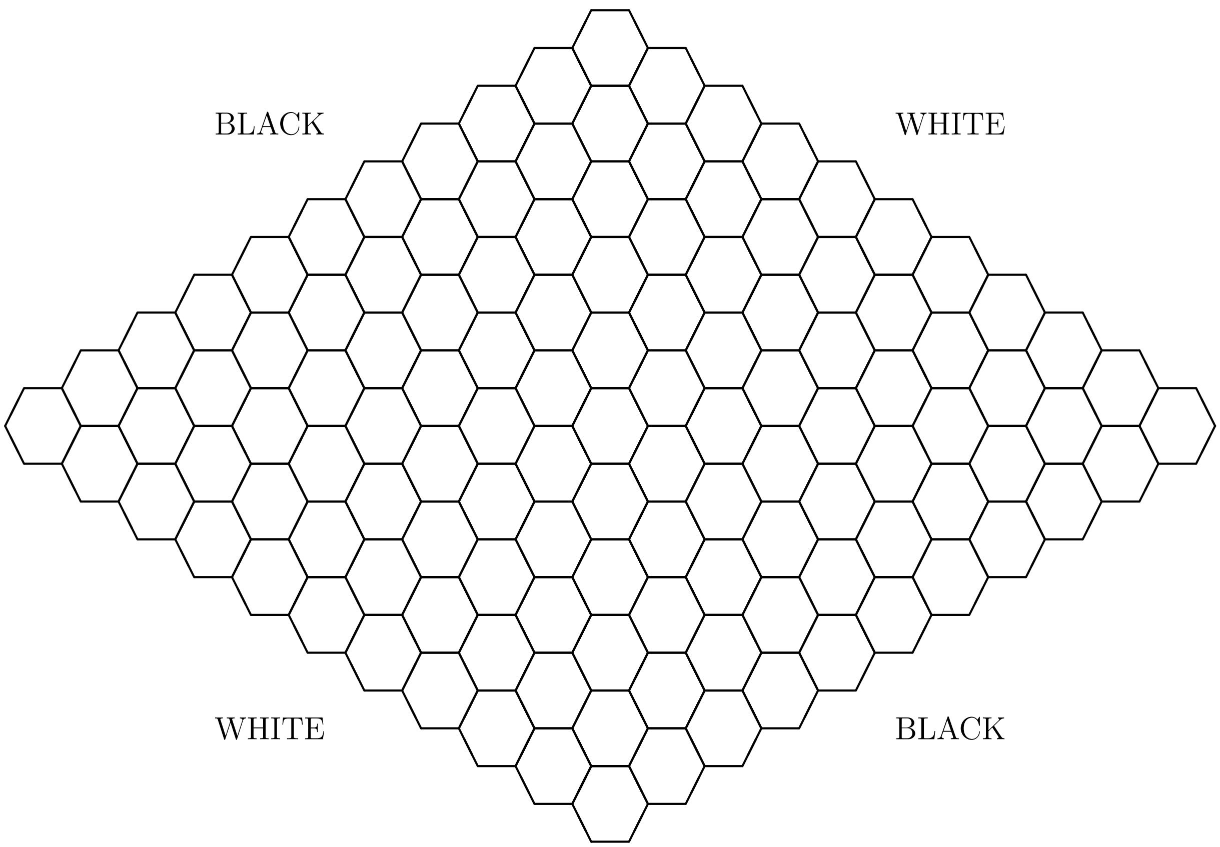



Figure 1: The Game of Hex

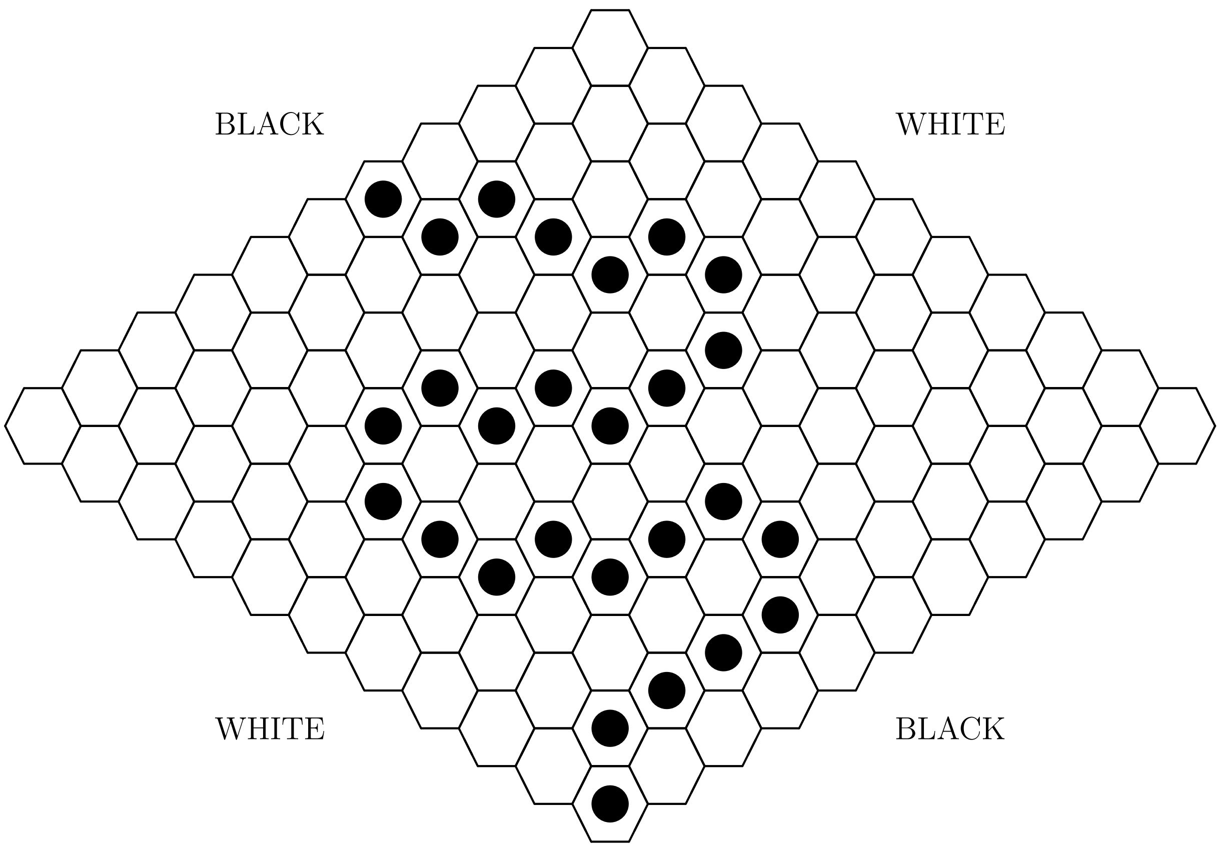



Figure 2: An Example of a Path in Hex

## 11 Hex II

Hex II is played on the board shown in Figure 3. Its rules are the same as Hex. Black is to play first.

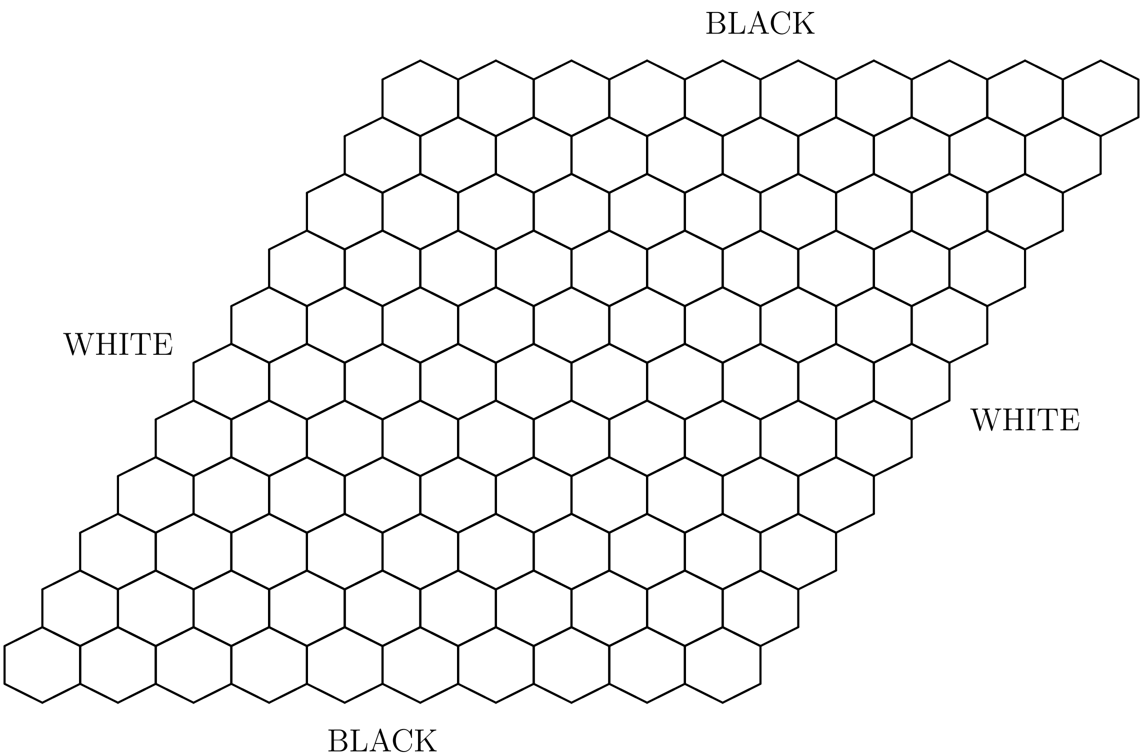



Figure 3: The Game of Hex II

## 12 Gale

A Gale board is shown in Figure 4. (Imagine that the hollow circles are actually red.) If it is played on paper, one player uses a black pencil for drawing a straight line to connect any pair of adjacent black dots, horizontally or vertically but not diagonally. the other player uses a red pencil for similarly joining pairs of red dots. Players take turns drawing lines. No line can cross another. The winner is the first player to form a connected path joining the two opposite sides of the board that are his color. Figure 5 shows the result of a game in which red has won.

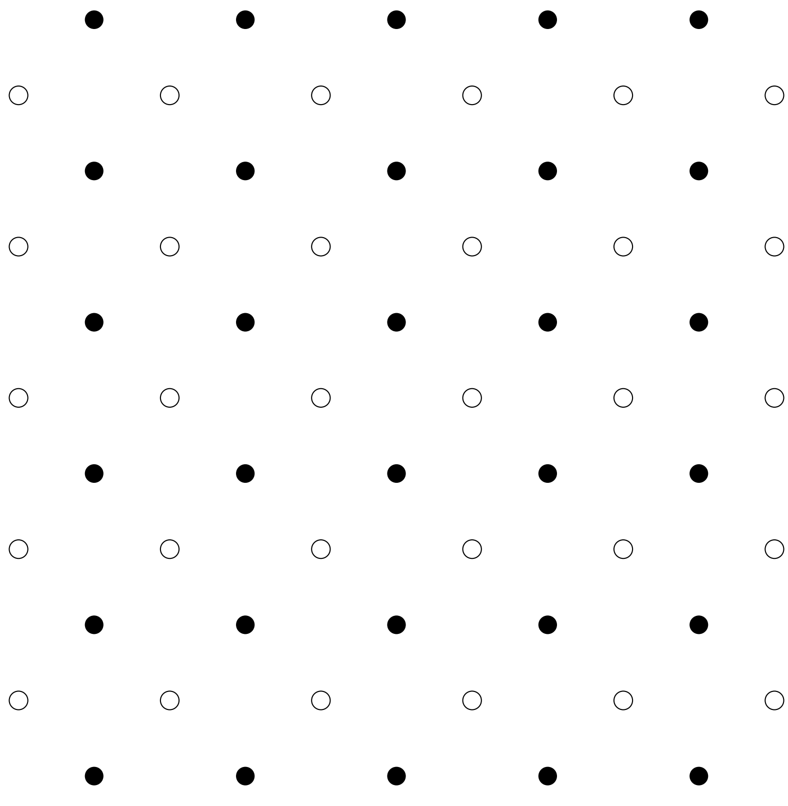


Figure 4: The Game of Gale

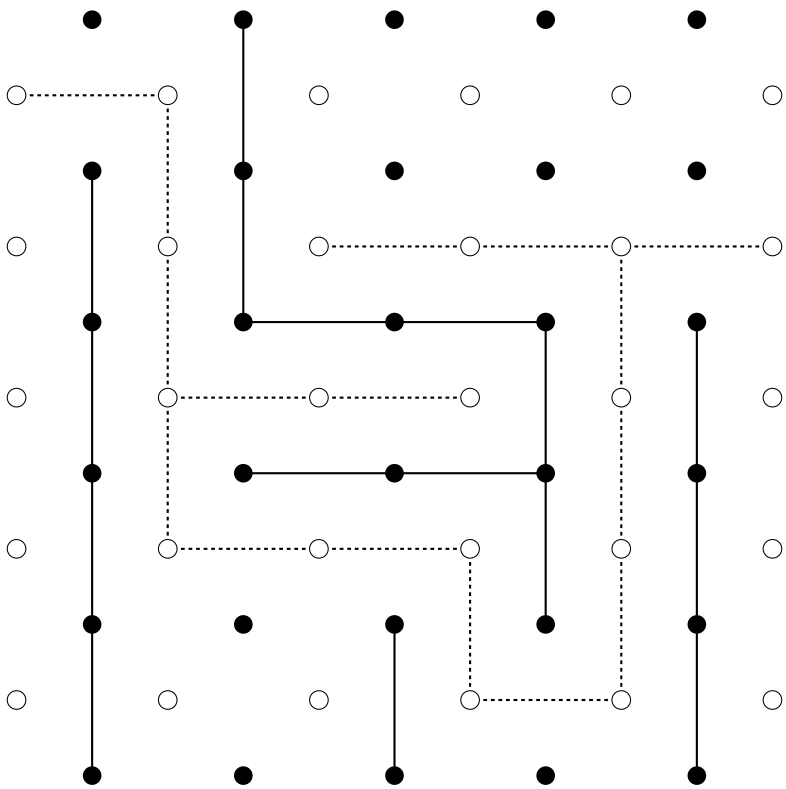
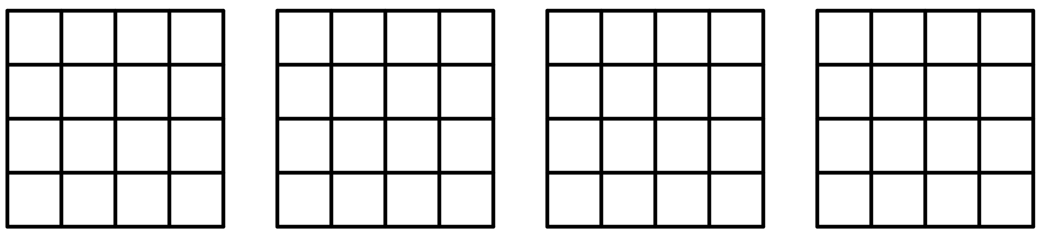


Figure 5: A Gale Game in which Red has Won

# 13 Relatives of Tic-Tac-Toe

## 13.1 Three-Dimensional Tic-Tac-Toe

This is familiar to most people. It is played on a $3 \times 3 \times 3$ board with the object of getting three in a row, or on a $4 \times 4 \times 4$ board with the object of getting four in a row. What is the optimal strategy?



## 13.2 Wild Tic-Tac-Toe

This is the same as tic-tac-toe, except that on your turn you may place either an $X$ or an $O$—your choice—in an empty cell. If this results in three-in-a-row with either symbol, then you win. Try this also with a $3 \times 3 \times 3$ board.

## 13.3 Toe-Tac-Tic

This is the same as tic-tac-toe, except that the first player to get three in a row *loses.* Try this also with a $3 \times 3 \times 3$ board.

## 13.4 Wild Toe-Tac-Tic

This is the same as wild tic-tac-toe, except that the first player to get three in a row *loses.* Try this also with a $3 \times 3 \times 3$ board.

### 13.5 Four-Dimensional Tic-Tac-Toe

Four-dimensional tic-tac-toe can be played on an imaginary hypercube by sectioning it into two-dimensional squares. A $4 \times 4 \times 4 \times 4$ hypercube, for example, would be diagrammed as shown in Figure 6.

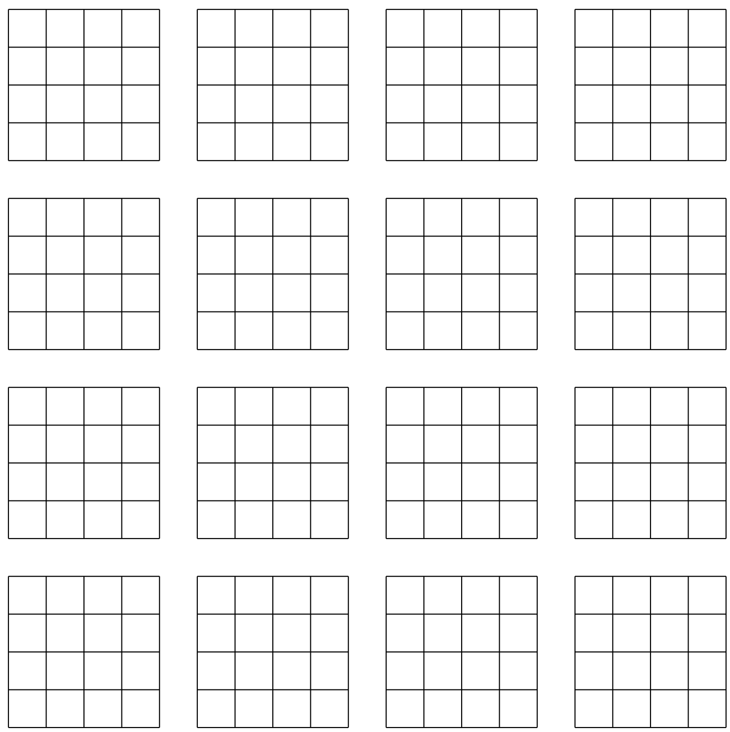


Figure 6: Four-Dimensional Tic-Tac-Toe

On this board a win of four in a row is achieved if four marks are in a straight line on any cube that can be formed by assembling four squares in serial order along any orthogonal or either of the two main diagonals. Figure 7 shows five examples of winning configurations. For example, if you occupy the four cells labeled 2, you win.

You can extend constructions of this type to play tic-tac-toe of any dimension!

## 14 Thinking about Thinking

This is more of a puzzle than a game, but you might try this first with different combinations of hats for the three people.

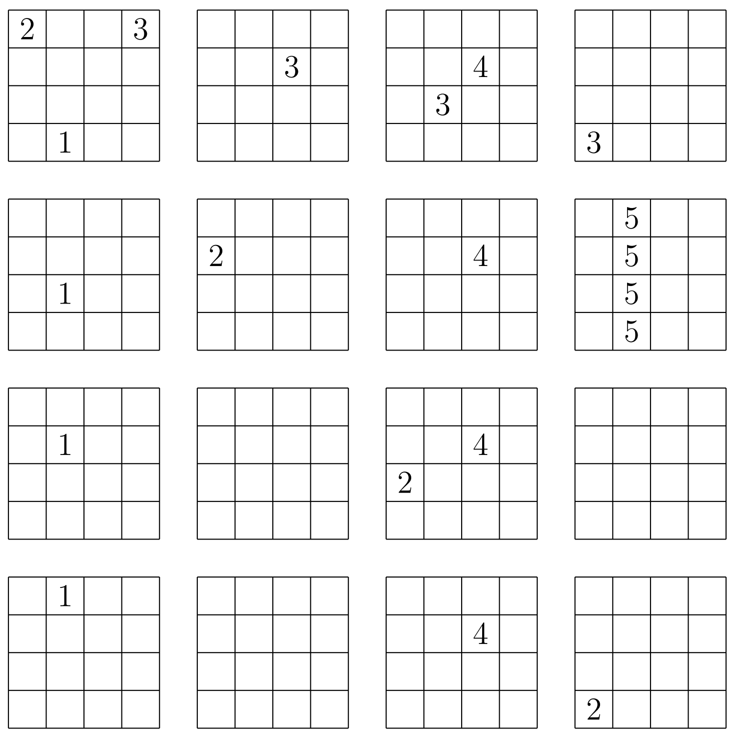


Figure 7: Four-Dimensional Tic-Tac-Toe

Three students — Alfred, Beth, and Carla — are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the students are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three students raise a hand. Several minutes go by until Carla, who is more astute than the others, leaves the room. How did she deduce the color of her hat?

## 15 More Thinking about Thinking

Two students, $A$ and $B$, are chosen from a math class of highly logical individuals. They are each given one positive integer. Each knows his/her own number, and is trying to determine the other's number. They are informed that their numbers are consecutive. In each of the following scenarios, what can you deduce about the two numbers?

1. First Scenario

   **A:** I know your number.

   **B:** I know your number.

2. Second Scenario

   **A:** I don't know your number.

   **B:** I know your number.

   **A:** I know your number.

3. Third Scenario

   **A:** I don't know your number.

   **B:** I don't know your number.

   **A:** I know your number.

   **B:** I know your number.

4. Fourth Scenario

   **A:** I don't know your number.

   **B:** I don't know your number.

   **A:** I don't know your number.

**B:** I don't know your number.

**A:** I know your number.

**B:** I know your number.

## 16 Morra

This game is played by two players, $R$ and $C$. Each player hides either one or two silver dollars in his/her hand. Simultaneously, each player guesses how many coins the other player is holding. If $R$ guesses correctly and $C$ does not, then $C$ pays $R$ an amount of money equal to the *total* number of dollars concealed by *both* players. If $C$ guesses correctly and $R$ does not, then $R$ pays $C$ an amount of money equal to the *total* number of dollars concealed by *both* players. If both players guess correctly or incorrectly, no money exchanges hands.

## 17 The Prisoner's Dilemma

From the Wikipedia article of this name (accessed 9/15/16): Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge. They hope to get both sentenced to a year in prison on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to: betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The offer is:

- If $A$ and $B$ each betray the other, each of them serves 2 years in prison
- If $A$ betrays $B$ but $B$ remains silent, $A$ will be set free and $B$ will serve 3 years in prison (and vice versa)
- If $A$ and $B$ both remain silent, both of them will only serve 1 year in prison (on the lesser charge).

See also `https://www.youtube.com/watch?v=_1SEXTVsxjk`

## 18 Set

This commercial game consists of a deck of 81 cards. Each card is made from one of three symbols, in one of three quantities, in one of three colors, and one of three shadings. Thus there are four attributes, and each card can be represented by an ordered 4-tuple $(a, b, c, d)$, where each of $a, b, c, d$ equals 1, 2, or 3. A set is a collection of three cards such that for each attribute they completely agree or completely disagree. Thus $(2, 3, 1, 1)$, $(2, 1, 1, 3)$, and $(2, 2, 1, 2)$ constitute a set.

An array of 12 cards is dealt, and players try to find sets. If a player detects a set, then he/she calls "set" and claims it, these three cards are given to the player, and three more cards are dealt into the vacant spots. If there are no sets, then three more cards are dealt to increase the size of the array. There is a penalty, say, of one set, if a player incorrectly calls "set" without identifying one. When the deck is exhausted, the player with the most sets is the winner. There is an official iPad app for this game that permits play for up to four players here: `https://itunes.apple.com/us/app/set-pro-hd/id381004916?mt=8`.

## 19 Sprouts

The game of Sprouts begins with $n$ spots on a sheet of paper. Even with as few as three spots, Sprouts is more difficult to analyze than tic-tac-toe, so that it is best for beginners to play with no more than three or four initial spots. A move consists of drawing a curve that joins one spot to another or to itself and then placing a new spot anywhere along the curve. These restrictions must be observed:

1. The curve may have any shape but it must not cross itself, cross a previously drawn curve or pass through a previously made spot.

2. No spot may have more than three curves emanating from it.

Players take turns drawing curves. In normal sprouts, the recommended form of play, the winner is the last person able to play.

## 20 Checkers, Chess, and Go

There has been tremendous progress in recent years on developing computer algorithms to play these games.

Checkers has been completely solved in the sense that there are unbeatable computer programs that show that under best play the game ends in a draw. See `http://www.scientificamerican.com/article/computers-solve-checkers-its-a-draw`.

On computer chess: `https://en.wikipedia.org/wiki/Computer_chess`.

On computer go: `https://en.wikipedia.org/wiki/Computer_Go`.

# 21 Other Topics

- Kuhn's Poker
- Checker Stacks
- Cooperative games and associated values
- Kruskal Count trick
- The Monty Hall problem
- Optimal gas station selection
- Towers of Hanoi
- The Chinese rings puzzle
- The Perfect Shuffle Theorem
- Risk and dynamic programming
- The Gambler's Ruin
- Minecraft
- Games for research

# 22 Further Reading

For discussion and analyses of many, many games, see Winning Ways for your Mathematical Plays, by Berlekamp, Conway, and Guy.