1 TWO COIN MORRA

This game is played by two players, R and C. Each player hides either one or two silver dollars in his/her hand. Simultaneously, each player guesses how many coins the other player is holding. If R guesses correctly and C does not, then C pays R an amount of money equal to the *total* number of dollars concealed by *both* players. If C guesses correctly and R does not, then R pays C an amount of money equal to the *total* number of dollars concealed by *both* players. If both players guess correctly or incorrectly, no money exchanges hands.

Clearly, each player must decide how many coins to hide and what number to guess. We will use the notation (1,2) to mean that a player hides 1 dollar and guesses "2." We can represent the possible outcomes of a round of play by a *payoff* matrix, indicating how much R will receive from C given the strategies followed by each player. A negative number means that C receives money from R.

		C			
		(1,1)	(1,2)	(2,1)	(2,2)
	(1,1)	0	2	−3	0
R	(1,2)	−2	0	0	3
	(2,1)	3	0	0	−4
	(2,2)	0	−3	4	0

This is an example of a *finite two-person zero-sum game.* "Finite" refers to the fact that each player has a finite number of strategies. "Zero-sum" refers to the fact that what one player gains in wealth, the other loses.

Suppose R decides to use only strategy (1,2). This is an example of a *pure strategy.* Since the minimum entry in that row is -2, R can guarantee losing no more than \$2 per round, and this will happen if C consistently uses strategy (1,1). But what if R decides to use either strategy (1,2) or (2,1), each half of the time, but randomly mixed so that there is no detectable pattern? For example, he could flip a coin to decide which of the two strategies to use. This is an example of a *mixed strategy.* To calculate the expected outcome, we mix the second and third rows by multiplying them each by 1/2 and adding them together. The result is

$$\begin{array}{c} \frac{1}{2}\left[\begin{array}{cccc} -2 & 0 & 0 & 3 \end{array}\right] \\ + \\ \frac{1}{2}\left[\begin{array}{cccc} 3 & 0 & 0 & -4 \end{array}\right] \\ \| \\ \left[\begin{array}{cccc} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{array}\right]. \end{array}$$

Since the minimum entry is $-1/2$, R can expect to lose no more than half a dollar per round on the average, and he can expect this to happen if C consistently chooses strategy (2,2).

Can R do better than this? If R uses each strategy $1/4$ of the time, then R mixes his rows by multiplying each of them by $1/4$ and adding them together, giving

$$\begin{array}{c} \frac{1}{4}\left[\begin{array}{cccc} 0 & 2 & -3 & 0 \end{array}\right] \\ + \\ \frac{1}{4}\left[\begin{array}{cccc} -2 & 0 & 0 & 3 \end{array}\right] \\ + \\ \frac{1}{4}\left[\begin{array}{cccc} 3 & 0 & 0 & -4 \end{array}\right] \\ + \\ \frac{1}{4}\left[\begin{array}{cccc} 0 & -3 & 4 & 0 \end{array}\right] \\ \| \\ \left[\begin{array}{cccc} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{array}\right]. \end{array}$$

Since the minimum entry is $-1/4$, R would expect to lose no more than a quarter dollar per round on the average, and he can expect this to happen if C consistently uses only strategies (1,2) and (2,2).

Problem 1.1 Try to find a mixed strategy for R which is even better. In particular, try to find nonnegative numbers p_1, p_2, p_3, p_4 that sum to one such that the minimum entry in

$$\begin{array}{c} p_1\begin{bmatrix} 0 & 2 & -3 & 0 \end{bmatrix} \\ + \\ p_2\begin{bmatrix} -2 & 0 & 0 & 3 \end{bmatrix} \\ + \\ p_3\begin{bmatrix} 3 & 0 & 0 & -4 \end{bmatrix} \\ + \\ p_4\begin{bmatrix} 0 & -3 & 4 & 0 \end{bmatrix} \end{array}$$

is at least zero. Can you find a mixture where the minimum entry is larger than zero?

C's strategies can be studied in a similar manner. For example, suppose C decides to use only strategy (2,2). The numbers in the matrix represent amounts that C pays R, so C is interested in looking at the maximum entry to see how badly he will do. Since the maximum entry in column 4 is 3, C can expect to lose at most \$3 per round, and this happens if R consistently uses strategy (1,2). If C uses each of his strategies a fourth of the time, we must mix the columns by multiplying each of them by 1/4 and adding them together. This yields

$$\frac{1}{4}\begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} -3 \\ 0 \\ 0 \\ 4 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 0 \\ 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}.$$

Since the maximum entry is 1/4, by using this mixed strategy, C can expect to lose no more than a quarter dollar per round on the average, and he can expect this if R consistently uses only strategies (1,2) and (2,2).

Of course, by symmetry, we can expect the best strategy for R to be the same as the best strategy for C.

Problem 1.2 Consider the following matching game. Each player hides either a nickel or a dime. If the two coins match, R gets C's coin. If they don't match, C gets R's coin. What are the optimal strategies for each player? Can they each expect to break even on the average? (If so the game is called *fair*.)

2 SADDLEPOINTS

Consider the game with the following payoff matrix

		C		
		I	II	III
	I	4	2	3
R	II	−3	0	4
	III	5	−1	−2

This game has an interesting property. Determine the smallest entry for each row. The largest of these is 2, and it occurs in row 1. Now determine the largest entry for each column. The smallest of these is also equal to 2 and it occurs in column 2. Because these two numbers are equal, this game is said to have a *saddlepoint.*

		C			
		I	II	III	
	I	4	2	3	2
R	II	−3	0	4	−3
	III	5	−1	−2	−2
		5	2	4	

This suggests that the optimal strategy for R is to use only strategy I, and the optimal strategy for C is to use only strategy II. For if R uses only I, R can expect to win at least $2 per round, and if C uses only II, C can expect to lose at most $2 per round. In general, we have the following theorem.

Theorem 2.1 *If the row minimum m in some row r equals the column maximum M in some column c, then the optimal strategy for R is to use only strategy r, the optimal strategy for C is to use only strategy c, and under optimal play the amount that R can expect to receive on the average is $m = M$.*

We will establish this theorem by considering a sufficiently general example. Suppose we have a 3×4 matrix in which the maximum entry in row 2 is 5 and the minimum entry in column 3 is also 5.

		C				
		1	2	3	4	
	1	·	·	a	·	
R	2	b	c	d	e	5
	3	·	·	f	·	
				5		

Since 5 is the minimum entry in row 2, we know that $d \geq 5$. But since 5 is the maximum entry in column 3, we know also that $d \leq 5$. So $d = 5$. Suppose R uses only strategy 2. Since 5 is the minimum row entry, R can expect to win at least \$5 per round. Now suppose R tries to do better by mixing his strategies according to p_1, p_2, p_3, where $p_1, p_2, p_3 \geq 0$, and $p_1 + p_2 + p_3 = 1$. Multiply the three rows by these numbers and add them together. Look at the third entry. It is $p_1 a + p_2 d + p_3 f$. Since 5 is the column maximum, $a \leq 5$ and $f \leq 5$. Remember that $d = 5$. Since $p_1, p_2, p_3 \geq 0$, we conclude $p_1 a \leq 5p_1$, $p_2 d = 5p_2$, and $p_3 f \leq 5p_3$. So $p_1 a + p_2 d + p_3 f \leq 5p_1 + 5p_2 + 5p_3 = 5(p_1 + p_2 + p_3) = 5(1) = 5$. Thus the third entry of the row mixture is no greater than 5, and hence the minimum entry of the row mixture is also no greater than 5. Therefore R cannot expect to win more than \$5 per round on the average even if he mixes his strategies.

A similar argument works for C. Suppose C uses only strategy 3. Since 5 is the maximum column entry, C can expect to lose at most \$5 per round. Now suppose C tries to do better by mixing his strategies according to q_1, q_2, q_3, q_4, where $q_1, q_2, q_3, q_4 \geq 0$ and $q_1 + q_2 + q_3 + q_4 = 1$. Multiply the four columns by these numbers and add them together. Look at the second entry. It is $q_1 b + q_2 c + q_3 d + q_4 e$. Since 5 is the row minimum, $b, c, e \geq 5$. Remember that $d = 5$. Since $q_1, q_2, q_3, q_4 \geq 0$, we conclude $q_1 b \geq 5q_1$, $q_2 c \geq 5q_2$, $q_3 d = 5q_3$, and $q_4 e \geq 5q_4$. So $q_1 b + q_2 c + q_3 d + q_4 e \geq 5q_1 + 5q_2 + 5q_3 + 5q_4 = 5(q_1 + q_2 + q_3 + q_4) = 5(1) = 5$. Thus the second entry of the column mixture is at least 5, and hence the maximum entry of the column mixture is also at least 5. Therefore C cannot expect to lose less than \$5 per round on the average even if he mixes his strategies.

3 MATRIX NOTATION

Given a matrix game with payoff matrix $A = (a_{ij})$. Suppose R uses mixed strategy $(p_1, \ldots, p_m)$ where $p_1, \ldots, p_m \geq 0$, $p_1 + \cdots + p_m = 1$, and C uses mixed strategy $(q_1, \ldots, q_n)$ where $q_1, \ldots, q_n \geq 0$, $q_1 + \cdots + q_n = 1$. Then R can expect to receive $\sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j$ per round on the average. This can be seen in two ways. If R mixes the rows according to his strategy, the resulting row is

$$\begin{aligned} & [p_1 a_{11} + \cdots + p_m a_{m1} \quad \cdots \quad p_1 a_{1n} + \cdots + p_m a_{mn}] \\ = \quad & [\textstyle\sum_{i=1}^m p_i a_{i1} \cdots \sum_{i=1}^m p_i a_{in}]. \end{aligned}$$

But C mixes his strategies according to $q_1, \ldots, q_n$, so the expected amount R wins per round on the average will be

$$\begin{aligned} & (\textstyle\sum_{i=1}^m p_i a_{i1}) q_1 + \cdots + (\sum_{i=1}^m p_i a_{in}) q_n \\ = \quad & \textstyle\sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j. \end{aligned}$$

The second way to see this is, if C mixes the columns according to his strategy, the resulting column is

$$\begin{bmatrix} q_1 a_{11} + \cdots + q_n a_{1n} \\ \vdots \\ q_1 a_{m1} + \cdots + q_n a_{mn} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j} q_j \\ \vdots \\ \sum_{j=1}^n a_{mj} q_j \end{bmatrix}$$

But R mixes his strategies according to $p_1, \ldots, p_m$, so the expected amount that R wins per round on the average will be

$$\begin{aligned} & p_1 (\textstyle\sum_{j=1}^n a_{ij} q_j) + \cdots + p_m (\sum_{j=1}^n a_{mj} q_j) \\ = \quad & \textstyle\sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j. \end{aligned}$$

This is more easily expressed using matrix multiplication. Write R's mixed strategy as a $1 \times m$ matrix

$$p = [p_1 \cdots p_m]$$

and C's mixed strategy as an $n \times 1$ matrix

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

Then if R mixes rows according to his strategy, the resulting row is pA. Now if C mixes columns according to his strategy, the result is an expected payoff of $(pA)q = pAq$ to R per round on the average.

Similarly, if C mixes columns according to his strategy, the resulting column is Aq. Now if R mixes rows according to his strategy, the result is an expected payoff of $p(Aq) = pAq$ to R per round on the average.

4 OPTIMAL STRATEGIES

If R uses mixed strategy p, then R can expect to win at least the minimum of the row pA per round on the average. Suppose this minimum is the cth entry and equals w. Then $w = pAq$ where

$$q = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

with a 1 in entry c and 0's everywhere else. Now consider any other

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

where $q_1, \ldots, q_n \geq 0$ and $q_1 + \cdots + q_n = 1$. Then $pA \geq [w \cdots w]$ so

$$\begin{aligned} (pA)q &\geq [w \cdots w]q \\ &= wq_1 + \cdots + wq_n \\ &= w(q_1 + \cdots + q_n) \\ &= w. \end{aligned}$$

We have shown that $(pA)q \geq w$ for all strategies q. So $w = \min_q (pA)q$. So R can expect to win at least $\min_q pAq$. Obviously R wants to choose p to make this minimum as large as possible, so R's goal is to find

$$\max_p \min_q pAq.$$

Starting all over again: If C uses mixed strategy q, then C can expect to lose at most the maximum of the column Aq. Suppose this maximum is the rth entry and equals w. Then $w = pAq$ where $p = [0 \cdots 1 \cdots 0]$ with a 1 in entry r and 0's everywhere else. Now consider any other $p = [p_1 \cdots p_m]$ where $p_1, \ldots, p_m \geq 0$ and $p_1 + \cdots + p_m = 1$. Then

$$Aq \leq \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix}$$

so

$$\begin{aligned} p(Aq) &\leq p \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix} \\ &= p_1 w + \cdots + p_m w \\ &= (p_1 + \cdots + p_m) w \\ &= w. \end{aligned}$$

We have shown that $p(Aq) \leq w$ for all strategies p. So $w = \max_p p(Aq)$. So C can expect to lose at most $\max_p pAq$.

Obviously C want to choose q to make this maximum as small as possible, so C's goal is to find

$$\min_q \max_p pAq.$$

Theorem 4.1 $\max_p \min_q pAq \leq \min_q \max_p pAq$, *where* $p_1, \ldots, p_m \geq 0$, $p_1 + \cdots + p_m = 1$, $q_1, \ldots, q_n \geq 0$, $q_1 + \cdots + q_n = 1$.

Proof: Consider any $\overline{p}$ and $\overline{q}$. Suppose $\min_q \overline{p}Aq$ occurs when $q = q^*$. Suppose $\max_p pA\overline{q}$ occurs when $p = p^*$. Then $\min_q \overline{p}Aq = \overline{p}Aq^* \leq \overline{p}A\overline{q} \leq p^*A\overline{q} = \max_p pA\overline{q}$. So no matter what $\overline{p}$ is chosen and what $\overline{q}$ is chosen, the left-hand side is always no more than the right-hand side. So even the largest left-hand side is no more than the smallest right-hand side. Hence $\max_p \min_q pAq \leq \min_q \max_p pAq$. $\square$

Theorem 4.2 *If* p^* *and* q^* *can be found so that*

$$\min_q p^*Aq = \max_p pAq^*,$$

then p^* *is optimal for* R *and* q^* *is optimal for* C *(where* p^* *and* q^* *are each nonnegative vectors whose components sum to 1).*

Proof: If there were a better $\overline{p}$, then

$$\min_q \overline{p}Aq > \min_q p^*Aq = \max_p pAq^*.$$

So

$$\max_p \min_q pAq \geq \min_q \overline{p}Aq > \max_p pAq^* \geq \min_q \max_p pAq$$

which contradicts the previous theorem.

Similarly, if there were a better $\overline{q}$, then

$$\min_q p^*Aq = \max_p pAq^* > \max_p pA\overline{q}.$$

So

$$\max_p \min_q pAq \geq \min_q p^*Aq > \max_p pA\overline{q} \geq \min_q \max_p pAq$$

which contradicts the previous theorem. □

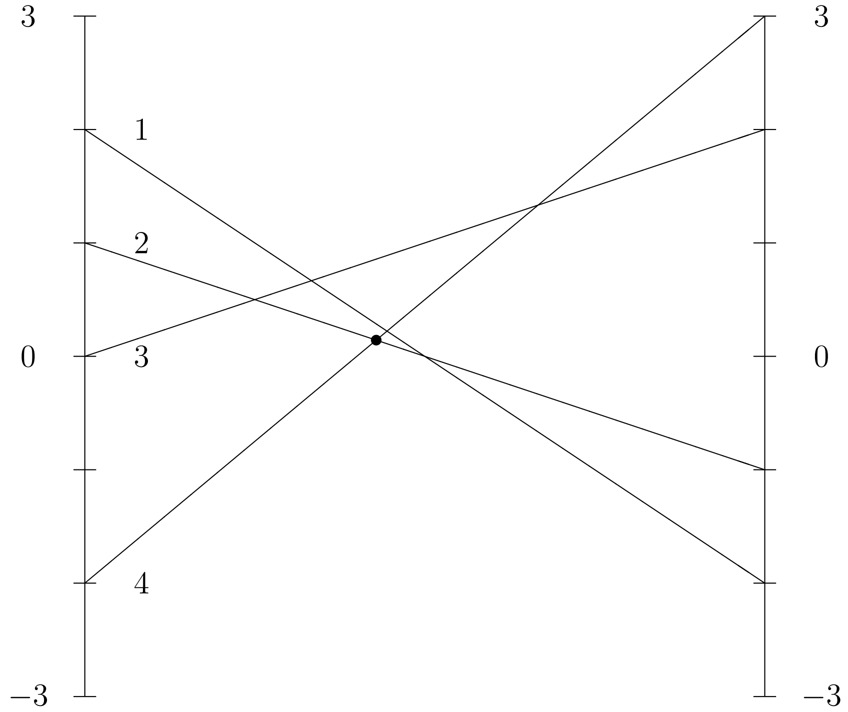
The above theorem helps to verify that proposed mixed strategies are optimal, although it does not indicate how to find them in the first place.

5 THE GRAPHICAL SOLUTION OF $2 \times n$ GAMES

If R has just two pure strategies, there is an easy way to determine R's optimal mix graphically. For example, consider the game

		C			
		1	2	3	4
R	1	2	1	0	−2
	2	−2	−1	2	3

Draw two parallel vertical axes. For each column, plot the first entry on the first axis and the second entry on the second axis. Connect the two points by a line segment and label the segment by the number of the column. Now darken the line segments which bound the figure from *below;* then find and mark with a dot the *highest point* on this darkened boundary.



The height of the dot as measured against the vertical axes is the value of the game. In the above example, the value of the game is 1/7. The position of the dot between the two axes indicates which mixture of his two strategies R should use. For example, if the dot is 3/7 of the way from the first axis to the second, as it is above, R should use his second strategy 3/7 of the time and his first strategy 4/7 of the time. The lines which intersect at the dot identify the strategies C should use in his mixture. In the above example, C should use only strategies 2 and 4.

In the above example, the dot is at the intersection of lines 2 and 4. This means that R should choose p_1 and p_2 so that in the row mixture, the 2nd and 4th entries turn out to be equal. Hence

$$\begin{gathered} p_1 - p_2 = -2p_1 + 3p_2 \\ 3p_1 = 4p_2 \\ p_1/p_2 = 4/3 \\ p_1 = 4/7 \quad p_2 = 3/7. \end{gathered}$$

The value of the game is $p_1 - p_2 = -2p_1 + 3p_2 = 1/7$. C should use only strategies 2 and 4 and should choose q_2 and q_4 so that in the column mixture, the two column entries turn out to be equal. Hence

$$q_2 - 2q_4 = -q_2 + 3q_4$$
$$2q_2 = 5q_4$$
$$q_2/q_4 = 5/2$$
$$q_2 = 5/7 \quad q_4 = 2/7.$$

Again, the value of the game is $q_1 - 2q_2 = -q_1 + 3q_4 = 1/7$. So by the last theorem of the previous section, we have found the optimal strategies for both players.

The above method can be modified to solve $m \times 2$ games as well, but then we have to interchange the role of the columns and the rows, darken the line segments which bound the graph from *above,* and locate the *lowest point* on this boundary.