SketchUp and Transformations

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CCSSM Content Standards on Transformations. The Common Core State Standards for Mathematics (CCSSM) places a central emphasis on rigid motions and similarity transformations to approach congruence and similarity. Focused encounters with such transformations occur in the middle grades. "Students use ideas about distance and angles, how they behave under translations, rotations, reflections, and dilations, and ideas about congruence and similarity to describe and analyze two-dimensional figures and to solve problems" (CCSSM). High school students must then build upon this foundation. "The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shapes generally)…. Similarity transformations (rigid motions followed by dilations) define similarity in the same way that rigid motions define congruence, thereby formalizing the similarity ideas of `same shape' and `scale factor' developed in the middle grades" (CCSSM).

CCSSM Mathematical Practice Standard on Modeling. The CCSSM also recognizes the key role of mathematical modeling in the learning of mathematics. It establishes modeling as one of the eight Standards for Mathematical Practice, explicitly weaving appropriate settings for modeling throughout all grade levels. "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace…. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose" (CCSSM).

The process of mathematical modeling rises in prominence at the high school level as a co-equal in value with content courses. The CCSSM states that modeling should be consciously incorporated into all of the courses at this level. "Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards…" (CCSSM).

Of course, mathematical modeling is much broader than just making physical or virtual models of geometrical objects. But this is certainly one important aspect of modeling, with a long, rich tradition in mathematics and mathematical visualization, and modeling with geometry offers powerful opportunities to bring geometric concepts to bear on applications in our everyday lives. For example, the high school geometry standards include modeling with geometry (G-MG), including "use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder)", "apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot)", and "apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios)" (CCSSM).

SketchUp Overview. SketchUp is well suited to simultaneously address the CCSSM transformations and modeling standards, as well as the engineering standards. It is a 3D geometric modeler powered by a rich set of geometric transformation tools. SketchUp Make is freely available from http://www.sketchup.com. It is focused on practical real world design, but also readily lends itself to both 2D and 3D mathematical constructions and explorations. At its heart it is a 3D modeling environment that can be populated by complex shapes created by the user from basic elements through a sequence of transformations. In addition, there is a vast library of pre-constructed models (in the "3D Warehouse") that the user can import and incorporate into a design. In addition, SketchUp offers the option of exporting models in file formats suitable for 3D printing. Figure 1 shows a puzzle assembled from 12 individually printed pieces.

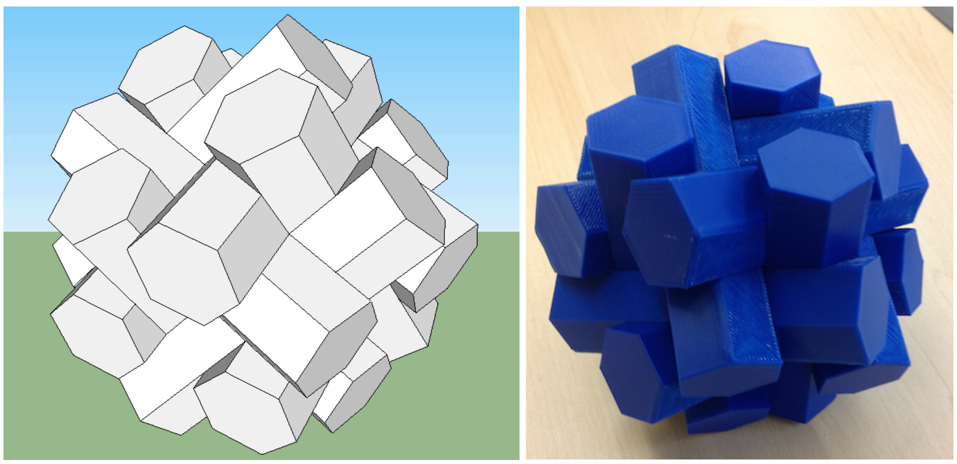

Figure 1

3D Printed Puzzle Created in SketchUp

A considerable collection of tutorials and educational resources for SketchUp already exists, some available directly from the SketchUp website, with many contributors at all levels of K-12 and college experience.

Environment. SketchUp operates within a 3D work environment, equipped with a set of axes. The user can use tools such as Orbit, Pan, Zoom, and Walk to change the observer's viewpoint, even to the extent of entering and moving around in spaces (such as rooms) in the design. Further, by setting the precise location and time of day on earth, the program can correctly compute and display the location of shadows. One can place individual elements of a design onto different "layers," and these layers can be selected and deselected for viewing in various combinations. For example, by placing the basic physical design of a classroom on one layer, and its furnishings (such as desks and chairs) on a second layer, it becomes easy to view the classroom with or without its furnishings.

Elements. The basic elements of SketchUp are lines and curves, two-dimensional shapes bounded by lines and curves, and three-dimensional objects composed of two-dimensional shapes. The designer can embellish the shapes with various colors and textures.

Tools for lines and curves include Line (for straight line segments), Freehand (for general curves), Arc and 2-Point Arc (for circular arcs). SketchUp offers various means of control over locations of points (for example, by exact or relative coordinates, by key points of previously constructed shapes, and by establishing guide lines) and locations of segments (by length or coordinates). When these elements enclose a planar region, the program will then create an inferred 2D shape.

For drawing 2D shapes the designer can use Rectangle (including squares and golden rectangles), Circle, Polygon (for regular polygons with specified numbers of sides), and Pie (for segments of circles). Again, there is considerable control of locations of points and sizes of objects.

The real power of SketchUp becomes immediately apparent with its set of transformation tools to construct ever more complicated three-dimensional shapes from simpler elements, as described below.

Transformations. Before describing the transformations it is helpful to understand that a subset of elements of a design (for example, a set of elements forming a chair, or the side of a box) may be fused together into a "group" or a "component." Then transformations can act on these groups or components as single entities. The primary difference between a group and a component is that if a component is replicated by, say, moving or rotating it, the new copies act as a set "clones" in the sense that any further editing of any one of the components will immediately be propagated to each clone. This is very convenient in modeling. For example, if one is designing a set of chairs, modifications and improvements to one chair apply to all of them automatically.

Move Tool. With this tool one can move or copy an object from one location to another. Thus one can effect the *translation* rigid motion. SketchUp offers several ways to determine the direction and distance of the translation, and the user can also create a specified number of multiple copies simultaneously. Note that if the user moves only part of given shape, the results can be interesting and perhaps unexpected. For example, starting with a rectangle, not fused into a group or component, and moving one side along itself using the Auto-Fold option, results in a shearing transformation producing a parallelogram . In a similar way one can construct general parallelepipeds by shearing rectangular parallelpipeds.

Rotate Tool. To *rotate* an object in three dimensions with the Move tool the user selects an axis, a starting reference point, and an ending reference point. Again, there are some options for specifying the extent of the rotation, and the object can be either rotated or copied into the new position.

Scale Tool. Use this tool to carry out *dilations* and related transformations. The designer can scale an object uniformly in all directions, or by independent scaling factors in all three axes directions. Though the user does not have precise control over the location of the point of dilation, the resulting object can subsequently be translated to a desired position. The Scale tool does not offer the option of making a scaled copy of the original object—leaving the starting object intact—but one can easily first use the Move tool to make a copy, and then apply the Scale tool to that copy.

2D Simulation. The Move and Rotate tools are, of course, designed for a 3D environment. But for pedagogical purposes a teacher may wish to consider the power of these tools when limited to 2D. One can simulate this in SketchUp by selecting the Parallel Perspective, Top Camera View and restricting the design to two-dimensional shapes. As mentioned before, it can be helpful to create "groups" of some of these shapes before acting upon them. It is now easy to work with the rigid motions in the plane.

Use the Move tool to carry out *translations* in a very natural way. For *rotations* select the axis by selecting a point; the actual axis is pointing up at the observer vertically, but this will not be evident. *Reflections* are more interesting: one can use the Rotate tool and select the axis of rotation by clicking and dragging along the desired line of reflection. Moving a reference point of the figure to the other side of the line results in the reflection of the object. But what is actually happening? Though it will not be obvious from this viewpoint, the object is rotating through the third dimension and coming to rest back in the plane. This is the traditional "flip" view of reflections in the plane—invoking the presence of a third dimension to carry out the reflection continuously. Pedagogically this is perhaps not the best way to view or define a reflection in the plane. For example, how will one then define reflection across a plane? It is indeed possible to make reference to a "flip" through a fourth dimension, but this is an unnecessary complication at the K-12 level.

It should be noted that SketchUp also offers a Flip command, which will result in reflecting the object in place across a plane parallel to a selected coordinate plane. But one has less control over the location of the reflecting plane. Also, if the user applies the Scale tool along a single direction with a scaling factor of -1, the resulting object will be a reflection of the original.

Push/Pull Tool. Using this tool the designer can select any two-dimensional shape and "extrude" it in a perpendicular direction to create a three-dimensional solid. Mathematically one can regard this as a *(right) prism* transformation. Because this tool operates on arbitrary 2D shapes, even those that are already parts of solids, it is extremely useful as well as empowering.

Follow Me Tool. In some ways this tool is a generalization of the Push/Pull tool. The designer can use this tool to extrude a 2D shape along the path of any line or curve. In particular, by extruding a planar region along the path of a circle, one can generate a solid of revolution.

Some Additional Tools. This list of tools and capabilities is not exhaustive. There are numerous other tools to assist in 3D design and modeling, including those to measure and mark lengths and angles, to calculate area and surface area, to introduce cross-sectional planes for cut-away views, and to add text and images.

Examples of Exploring Mathematics with SketchUp. As design software grounded upon transformations, SketchUp is well positioned as a sophisticated, yet intuitive, technological tool to explore mathematical concepts of the CCSSM. Here is a brief, representative list of examples.

Describe how various objects can be defined by means of transformations. Regular polygon (appropriately rotate a radius of the polygon). Parallelogram (rotate a triangle about the midpoint of one of its sides or create a rectangle and sheer it by moving one of the sides). Ellipse (scale a circle in a single direction). Rectangular prism (extrude a prism over a rectangle). Cylinder (extrude a prism over a circle; construct the volume of revolution of a rectangle). Sphere (construct the volume of revolution of a circle about a perpendicular circle of the same center). Torus (construct the volume of revolution of a circle about a suitably sized perpendicular circle with a different center). Pyramid or cone (create a prism and then apply a scaling with small positive scaling factor (e.g., 0.001) to one of the bases only. Hyperboloid (e.g., power plant cooling tower) (create a circular cylinder and rotate just one of the bases).

Describe the symmetries of various objects by means of transformations. Isosceles triangles, regular polygons, rectangles, rhombi, kites, (mapped to themselves via certain rotations and reflections). Platonic solids, prisms (mapped to themselves via certain 3D rotations and reflections across planes).

Develop two-dimensional area formulas via dissection and transformation arguments (dissect the starting object, turn appropriate pieces into groups and then apply some combination of translations, rotations, and reflections.)

Develop surface area formulas for three-dimensional objects (for polyhedra use 3D rotations to unfold the faces into the plane to make nets).

Develop volume formulas for three-dimensional objects (approximate objects by piles of cubes; use cross-sectional planes to illustrate Cavalieri's principle to match volumes of rectangular and nonrectangular parallelepipeds, and to match the volume of a hemisphere with that of a cylinder with an inverted cone removed).

Motivate the effect of scaling on length, area, surface area, and volume (scale planar regions superimposed on grids of unit squares, and scale three-dimensional structures composed of unit cubes).

Illustrate congruence via combinations of rigid motions and similarity via combinations of scalings and rigid motions (given two planar figures, test congruence by attempting to move one to exactly match the other through a sequence of appropriate transformations).

Tesselations (create a tile as a component, replicate it multiple times, and move the copies around to tile the plane).

Illustrate the Pythagorean Theorem (make a model of one of the dissection proofs).

Construct and visualize solids through intersections and unions (construct two perpendicular cylinders with crossing axes, use the Intersect Faces command to compute all of the intersections of two dimensional elements, delete away the portions that are not in the common 3D intersection).