Calculus I
Bypass examination

UK Department of Mathematics
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Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer **(unsupported answers may not receive credit).** 3) give exact answers, rather than decimal approximations to the answer.
Please write out the work you used to find the solution and then put the answer in the space provided.

Name KEY
School
Calculus teacher

Question	Score	Total
1		6
2		6
3		6
4		6
5		10
6		6
7		8
8		10
9		10
10		10
11		12
12		10
		100

1. Find the derivatives of the following functions.

(a) $f(x) = \frac{1}{\sqrt{x^2+1}}$.

(b) $g(x) = \sin(x^2)$.

a) $f(x) = (x^2+1)^{-1/2}$

$f'(x) = \frac{-1}{2}(x^2+1)^{-3/2} 2x$

+3

$f'(x) = \frac{-x}{(x^2+1)^{3/2}}$

+3

b) $g(x) = \sin(x^2)$

$g'(x) = \cos(x^2) 2x$

$= 2x\cos(x^2)$

3 each points taken off for any errors.

(a) $\frac{-x}{(x^2+1)^{3/2}}$, (b) $2x\cos(x^2)$

2. Find the equation of the tangent line to the graph of $y = x^2 + 3x$ at the point $(-1, -2)$. Put your answer in the form $y = mx + b$.

Slope $y' = 2x + 3$ at $x = -1$

$2(-1) + 3 = 1$

+3

Point Slope $y - -2 = 1(x - -1)$

$y + 2 = 1(x+1)$

+2

$y = x + 1 - 2$

$y = x - 1$ +1

$y =$ $x - 1$

3. Suppose f and g are continuous functions whose domain is the interval $(0, 8)$ and we have

$$\lim_{x\to 4} f(x) = 3, \quad \lim_{x\to 5} f(x) = 7, \quad \lim_{x\to 4} g(x) = 5, \quad \text{and} \quad \lim_{x\to 5} g(x) = 2.$$

Find the limits.

(a) $\lim_{x\to 4}(f(x) + g(x))$
(b) $\lim_{x\to 5}(xf(x) + g(x))$
(c) $\lim_{x\to 4} f(g(x) - 1)$

+2 a) $\lim_{x\to 4} f(x)+g(x) = \lim_{x\to 4} f(x) + \lim_{x\to 4} g(x) = 3+5 = 8$

+2 b) $\lim_{x\to 5} xf(x)+g(x) = \lim_{x\to 5} xf(x) + \lim_{x\to 5} g(x) = 5(7)+2 = 37$

+2 c) $\lim_{x\to 4} f(g(x)-1) = f(\lim_{x\to 4} g(x) - 1) = f(5-1) = f(4)$ by continuity $= 3$

(a) 8, (b) 37 (c) 3

4. Let

$$f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{x - 2}, & x \neq 2 \\ A, & x = 2. \end{cases}$$

Find the value of A for which f will be continuous.

Take limit

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{(x-1)(x-2)}{x-2} = \lim_{x\to 2} x - 1 = 1$$ +3

To be continuous

$\lim_{x\to a} f(x) = f(a)$ hence $f(2) = 1$ +2

$A =$ 1 +1

5. (a) Give the definition of the derivative of a function f at a point a.

The derivative of a function f at a number a is

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{or} \quad \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

if the limit exists.

+4

2 pts off if not a complete sentance

(b) Use the definition to find $f'(3)$ for the function f defined by

$$f(x) = \frac{x}{x+1}.$$

+6

$$\lim_{h\to 0} \frac{\frac{3+h}{4+h} - \frac{3}{4}}{h} = \lim_{h\to 0} \frac{\frac{4(3+h)}{4(4+h)} - \frac{(4+h)(3)}{4(4+h)}}{h}$$

$$= \lim_{h\to 0} \frac{12+4h-12-3h}{4(4+h)h}$$

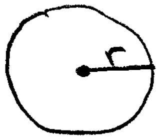
$$= \lim_{h\to 0} \frac{h}{4(4+h)h}$$

$$= \lim_{h\to 0} \frac{1}{4(4+h)} = \frac{1}{4(4)} = \frac{1}{16}$$

Answer $\frac{1}{16}$

6. The area of a circle is increasing at a rate of 3 meters^2/second. Find the rate of change of its radius with respect to time when the radius is 2 meters.

$$A = \pi r^2$$



3

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Use area of a circle formula and take derivative of both sides with respect to time

$$3\,\frac{m^2}{s} = 2\pi(2)\frac{dr}{dt}$$

3

$$\frac{3}{4\pi}\,m/s = \frac{dr}{dt}$$

plug in the appropriate value

$\frac{3}{4\pi}$ m/s

7. Let $f(x) = x^3 - 6x^2 + 20$.

(a) Find the critical numbers for f.

(b) Find the locations of local maxima and minima for f. Explain your answer.

(a) Take derivative and set equal to zero.

$$f'(x) = 3x^2 - 12x = 0$$

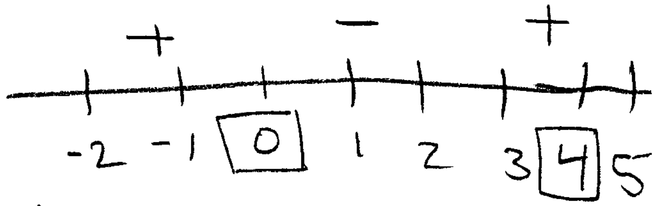
$$3x(x-4) = 0$$

+4

$\boxed{x=0}$ or $x-4=0$

$\boxed{x=4}$

+4 b) using first derivative test



Test values of derivative on either side.

(a) Critical numbers at $x =$ 0, 4

$f'(-1) = 15$

(b) Local maxima at $x =$ 0

$f'(1) = -9$

Local minima at $x =$ 4

$f'(5) = 15$

f increases on $(-\infty, 0)$

f decrease on $(0, 4)$

8. A particle moves along the x-axis so that at time t seconds it is $x(t) = -t^3 + 3t^2$ meters to the right of the origin.

(a) Find the position, velocity and acceleration of the particle at time $t = 2$ seconds.

(b) Is the particle moving to the right or the left at time $t = 4$ seconds?

(c) What is the farthest that the particle is from the origin for t in the interval $[0, 3]$?

a) $x(t) = -t^3 + 3t^2$ position $\quad x(2) = -(2)^3 + 3(2)^2 = -8 + 12 = 4$

+3 $\quad x'(t) = -3t^2 + 6t$ velocity $\quad x'(2) = -3(2)^2 + 6(2) = 0$

$x''(t) = -6t + 6$ acceleration $\quad x''(2) = -6(2) + 6 = -12 + 6 = -6$

Units? okay.

b) $x'(4) = -3(4)^2 + 6(4)$

$= -3 \cdot 16 + 24$

+2 $\quad = -48 + 24$

$= -24$ (negative) so it is moving to the left.

c) Look at derivative for max values and check endpoints

+4 $\quad -3t^2 + 6t = 0$

$3t(2 - t) = 0$

$t = 0$, $t = 2$ > critical points

$f(0) = 0$

$\boxed{f(2) = 4}$

$f(3) = -27 + 3(9) = 0$

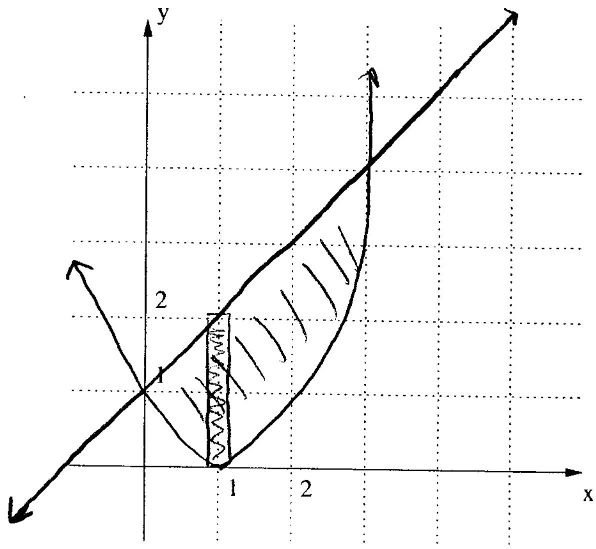
(a) Position 4 m, Velocity 0 m/s,

Acceleration -6 m/s^2

+1

(b) Left (c) 4 meters

9. Sketch the graphs of the curves defined by $y = x + 1$ and $y = (x-1)^2$. Find the area between these curves.



+3

Find where they intersect

+3

$x+1 = (x-1)^2 = x^2-2x+1$

$-x\ -1 \qquad -x\ -1$

$0 = x^2-3x$

$0 = x(x-3)$

$x=0$ or $x-3=0$

$x=3$

+4 $\displaystyle\int_0^3 (x+1)-(x-1)^2\,dx = \int_0^3 x+1-x^2+2x-1\,dx$

$= \displaystyle\int_0^3 -x^2+3x\,dx$

$= -\frac{1}{3}x^3+\frac{3}{2}x^2\Big|_0^3$

$= -9+\frac{27}{2}$

$= -\frac{18}{2}+\frac{27}{2} = 9/2$

The area is $9/2$

10. Evaluate the following definite integrals. Be sure to show all your work.

$$(a)\ \int_0^1 t\sqrt{1+4t^2}\,dt \qquad (b)\ \int_0^{\pi/2} \cos(t)\,\sin^2(t)\,dt$$

a) $u = 1+4t^2$ +2

$du = 8t\,dt$

$t=0$ then $u=1$

$t=1$ then $u=5$

$\displaystyle\frac{1}{8}\int_1^5 \sqrt{u}\,du = \frac{1}{8}\left(\frac{2u^{3/2}}{3}\Big|_1^5\right)$ +3

$= \frac{1}{48}\cancel{2}\frac{}{3}(5^{3/2}-1)$

$= \frac{1}{12}(5^{3/2}-1)$

b) $u = \sin t$ $\quad t=0, u=0$

2 $du = \cos t\,dt$ $\quad t=\pi/2, u=1$

$\displaystyle\int_0^1 u^2\,du = \frac{1}{3}u^3\Big|_0^1$

+3 $= \frac{1}{3}(1)^3 - 0 = \frac{1}{3}$

(a) $\frac{1}{12}(5^{3/2}-1)$, (b) $1/3$

(Note: It is not necessary to change bounds.)

11. A company wishes to manufacturer a box with a square base and a volume of 625 cubic centimeters. The material for the top and bottom costs \$0.10 per square centimeter and the material for the sides costs \$0.02 per square centimeter. Find the dimensions and cost of the cheapest such box.

Cost to make = surface area multiplied by the appropriate charges

+3

$\boxed{\text{Let } x \text{ be the length and width and } h \text{ the height}}$

$$\text{Cost} = \underbrace{2x^2(.10)}_{\text{top and bottom area multiplied by charge}} + \underbrace{4xh(.02)}_{\text{sides multipled by charge}}$$

+2

Note Square bottom and top but sides may not be.

We have volume

$V = 625 = x^2 h$

hence $h = \frac{625}{x^2}$

We want to minimize $C(x)$. We take derivative. Since $h = \frac{625}{x^2}$ we have

$$C(x) = \frac{2}{10}x^2 + 4x\left(\frac{625}{x^2}\right)\frac{2}{100}$$

$$= \frac{2}{10}x^2 + \frac{50}{x} = \frac{2}{10}x^2 + 50x^{-1}$$

$C'(x) = \frac{4}{10}x - 50x^{-2} = 0$ $\boxed{\text{Find C.P.}}$

$\frac{4}{10}x = \frac{50}{x^2}$

$\frac{4x^3}{4} = \frac{500}{4}$

$x^3 = 125$

$\boxed{x = 5}$

+2

+2

We have $x = 5$ as our C.P.

Check that it is a minimum

$$\begin{array}{cccccc} & - & & + & & \\ 0 & 3 & \boxed{5} & 7 & 10 & \end{array}$$

$\cup$ min.

$f'(1) = -49.6$

$f'(6) = 2.4 - \frac{50}{(6)^2} =$

① So $x = 5$ is our min value

$h = \frac{625}{25} = 25$

② $C(5) = \frac{2}{10}(5)^2 + \frac{50}{5} =$

$= 5 + 10 = \$15$

+3

Dimensions 5cm x 5cm x 25cm

Cost \$15

12. Let

$$F(x) = \int_1^x \frac{1}{2+\sin(t)}\,dt.$$

(a) Find the derivative $F'(x)$.

(b) Find the intervals where the function F is concave up or concave down.

a) use fundamental theorem of calculus (+4)

$$F'(x) = \frac{1}{2+\sin x} = (2+\sin x)^{-1}$$

b) Find where the second derivative is zero

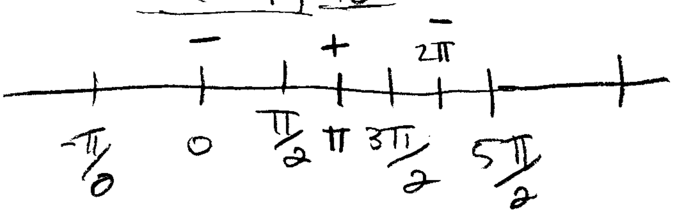
$$F''(x) = -1(2+\sin x)^{-2}(\cos x) \quad (+2)$$

$$= \frac{-\cos x}{(2+\sin x)^2}$$

$F''(x) = 0$ then $\cos x = 0$ hence when $x = (2n+1)\frac{\pi}{2}$

or odd multiples of $\frac{\pi}{2}$

use concavity test (+2)



$f''(0) = \frac{-1}{4}$ $\quad(-\pi/2, \pi/2)$ $f'' < 0$

$f''(\pi) = \frac{1}{4}$ $\quad(\pi/2, 3\pi/2)$ $f'' > 0$

$f''(2\pi) = \frac{-1}{4}$ $\quad(3\pi/2, 5\pi/2)$ $f'' < 0$

(a) $F'(x) =$ $\frac{1}{2+\sin x}$

(+2)

(b) Intervals where F is concave up $\left(\frac{\pi}{2}+2\pi k, \frac{3\pi}{2}+2k\pi\right)$ $k = 0, \pm 1, \pm 2, \ldots$

Intervals where F is concave down $\left(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right)$, $k = 0, \pm 1$