MA 113 — Calculus I
Exam 1

Fall 2008
September 23, 2008

Answer all of the questions 1 – 7 and <u>two</u> of the questions 8 – 10. Please indicate which problem is not to be graded by crossing through its number on the table below.

Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. **clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may not receive credit),**
2. **give exact answers, rather than decimal approximations.**

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Answer Key

Section: ________________

Last four digits of student identification number: __________

Question	Score	Total
1		9
2		7
3		11
4		12
5		11
6		9
7		6
8		16
9		16
10		16
Free	3	3
		100

(1) (a) Using logarithms solve the identity $5^{4x-10} = \frac{1}{25}$. **Show your work!**

3

$\log_5(5^{4x-10}) = \log_5\left(\frac{1}{25}\right) = -2$

b/c $5^{-2} = \frac{1}{25}$

$4x - 10 = -2$

$4x = 8$

$\boxed{x = 2}$

1

(b) Simplify the expression

$$\log_3(x) + \frac{1}{2}\log_3(5x) - \log_3(10+x)$$

such that it is a single logarithm.

5

$\log_3(x) + \frac{1}{2}\log_3(5x) - \log_3(10+x)$

$= \log_3(x) + \log_3\left((5x)^{\frac{1}{2}}\right) - \log_3(10+x)$

$= \log_3\left(\frac{x \cdot (5x)^{\frac{1}{2}}}{10+x}\right) = \log_3\left(\frac{x \cdot \sqrt{5x}}{10+x}\right)$

b) using the laws of logarithms.

partial credit

(a) $x =$ 2

(b) Expression is $\log_3\left(\frac{x \cdot \sqrt{5x}}{10+x}\right)$

(2) Suppose that $\lim_{x\to 3} f(x)$ exists and satisfies

$$\lim_{x\to 3}\left(\frac{1}{x+2}f(x)+3x-1\right)=14.$$

Use the limit laws to compute $\lim_{x\to 3} f(x)$. **Explain your reasoning.**

Put $L = \lim_{x\to 3} f(x)$. Using the limit laws we get

3

$$\underbrace{\lim_{x\to 3}\frac{1}{x+2}}_{\frac{1}{5}} \cdot \underbrace{\lim_{x\to 3} f(x)}_{L} + \underbrace{\lim_{x\to 3}(3x-1)}_{8} = 14$$

$$\frac{1}{5} \cdot L + 8 = 14$$

2

where we obtained those limits because rational functions and polynomials are continuous on their domain.

2

Now we solve

$$\frac{1}{5}\cdot L + 8 = 14$$

$$\frac{1}{5}\cdot L = 6$$

$$\boxed{L = 30}$$

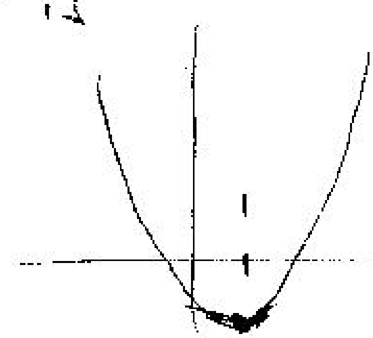
$\lim_{x\to 3} f(x) =$ 30

(3) Let $f(x) = x^2 - 2x - 8$.

(a) Find the maximal value for a such that f is one-to-one on the interval $(-\infty, a]$.

(2) Completing the square yields
$f(x) = (x-1)^2 - 9$,
hence the vertex of the parabola is at
1 and $f(1) = -9$. Then f is

(2) one-to-one on $(-\infty, 1]$



(b) For the value of a you found in part (a) find the inverse f^{-1} of the function f with the domain $(-\infty, a]$.

(1) $x^2 - 2x - 8 = y$

(1) $x^2 - 2x - 8 - y = 0$

(2) $x = 1 \pm \sqrt{1 + 8 + y}$

$x = 1 \pm \sqrt{9 + y}$

(2) Because the domain is $(-\infty, 1]$
we have to choose
$x = 1 - \sqrt{9+y}$.

(1) Thus $f^{-1}(x) = 1 - \sqrt{9+x}$.

(a) $a =$ 1

(b) $f^{-1}(x) =$ $1 - \sqrt{9+x}$

(4) A particle is traveling on a straight line so that its position after t seconds is given by $s(t) = 40t - 16t^2$ centimeters.

(a) Find the average velocity of the particle during the time interval $[1, 3]$.

avg. velocity $= \dfrac{\text{distance traveled}}{\text{time elapsed}}$

1 $= \dfrac{s(3)-s(1)}{3-1} = \dfrac{(120-144)-(40-16)}{2}$

2 $= \boxed{-24}$

(b) Find the average velocity of the particle during the time interval $[1, t]$, where $t > 1$. Simplify your answer.

2 $\dfrac{s(t)-s(1)}{t-1} = \dfrac{(40t-16t^2)-(40-16)}{t-1}$

2 $= \dfrac{-16t^2+40t-24}{t-1} = \dfrac{(t-1)(-16t+24)}{t-1}$

1 $\underset{t \neq 1}{=} \boxed{-16t+24}$

(c) Find the instantaneous velocity of the particle at time $t = 1$.

2 $\lim\limits_{t \to 1} \dfrac{s(t)-s(1)}{t-1} = \lim\limits_{t \to 1} (-16t+24) = 8$ 1 1

true for $t > 1$ as well as $t < 1$

continuity of polynomials

(a) average velocity on $[1, 3]$ is -24 cm/sec

(b) average velocity on $[1, t]$ is $-16t+24$ cm/sec

(c) instantaneous velocity at time $t = 1$ is 8 cm/sec

(5) Compute the following limits or show that they do not exist.

(a) $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} =$ $\underline{\;2/5\;}$

2 $\frac{x^2-4x+3}{x^2-x-6} = \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+2)} \underset{x\neq 3}{=} \frac{x-1}{x+2}$

1 Since $\frac{x-1}{x+2}$ is defined at 3 it is also continuous at 3 (being a rational function) and thus

1 $\lim_{x\to 3} \frac{x-1}{x+2} = \frac{2}{5}$

(b) $\lim_{x\to 3} \frac{x^2+x-8}{x^2-4x-5} =$ $\underline{\;-\frac{1}{2}\;}$

1 Since 3 is in the domain of this rational function, the function is continuous at 3 and the limit is

2 $f(3) = \frac{4}{-8} = -\frac{1}{2}$

(c) $\lim_{x\to 3} \frac{x^2+x+5}{x^2-4x+3} =$ $\underline{\;\text{DNE}\;}$

for $x\to 3$ → 17

2 $\frac{x^2+x+5}{x^2-4x+3} = \frac{x^2+x+5}{(x-3)(x-1)}$ for $x\to 3$ → 2

1 for $x<3$: $x-3<0$
for $x>3$: $x-3>0$, left hand side limit is $-\infty$, right one is $+\infty$.

1

(6) Let f be the function defined as

$$f(x) = \begin{cases} x^2 + \sqrt{2x-6}, & \text{for } x \geq 5 \\ x^2 + x - 3, & \text{for } x < 5 \end{cases}$$

Show that the function f is continuous at $x = 5$. **Clearly explain your reasoning.**

2 • $f(5) = 5^2 + \sqrt{2 \cdot 5 - 6} = \underline{27}$

2 • $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x^2 + x - 3) = \underline{27}$

continuity of polynomials

2 • $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x^2 + \sqrt{2x-6}) = 27$

1 $\Rightarrow$ continuity of root fct's, polynomials and because sums and composition of cont. fct's are cont. again.

1 Thus, $\lim_{x \to 5} f(x)$ exists and

$\lim_{x \to 5} f(x) = 27 = f(5)$.

(7) Let f be a function such that, for all real numbers x,

$$2x^3 - 3x^2 - 2 \le f(x) \le x^4 - 2x^3 + 4x^2 - 6x.$$

Show that $\lim_{x \to 1} f(x)$ exists and find its value. **As usual, justify your answer.**

1 $\lim_{x \to 1} (2x^3 - 3x^2 - 2) = 2 - 3 - 2 = -3$

continuity of polynomials

1 $\lim_{x \to 1} (x^4 - 2x^3 + 4x^2 - 6x) = 1 - 2 + 4 - 6 = -3$

2 thus, by squeeze theorem,

2 $\lim_{x \to 1} f(x)$ exists and has value -3

$\lim_{x \to 1} f(x) =$ -3

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front page of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

4

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

(b) Let b and c be numbers and

$$f(x) = \begin{cases} 2 - x^2 & \text{for } x < 1, \\ bx + c & \text{for } 1 \le x \le 3, \\ 10 - x & \text{for } x > 3. \end{cases}$$

4

Find

$\lim_{x \to 1^-} f(x) =$ 1, $\lim_{x \to 1^+} f(x) =$ $b+c$,

$\lim_{x \to 3^-} f(x) =$ $3b+c$, $\lim_{x \to 3^+} f(x) =$ 7.

Some of your answers involve b and c.

E.g. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (bx+c) = b+c$. direct substitution

(c) Find the values of b and c such that the function f in part (b) is continuous at all real numbers.

2

The function is continuous at all numbers except for 1 and 3 regardless of b and c. For continuity at 1 and 3

2

we need $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

and $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$.

?

Thus $1 = b + c$ and $3b + c = 7$. Solving

2

this system yields $b = 3$ and $c = -2$.

With these values the function is indeed continuous at all real numbers.

$b =$ 3 $c =$ -2

(9) (a) State the definition of the derivative of a function f at a number a. Use complete sentences. the derivative of f at a is defined

2 as $f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

2 if this limit exists.

(b) Find the slope of the secant line to the graph of the function

$$f(x) = x^2 - 3x$$

through the points $(a, f(a))$ and $(a+h, f(a+h))$, where $h \neq 0$. Simplify your answer.

2 $\frac{f(a+h) - f(a)}{a+h-a} = \frac{\cancel{a^2} + 2ah + h^2 - \cancel{3a} - 3h - \cancel{a^2} + \cancel{3a}}{h}$

2 $= \frac{h^2 + 2ah - 3h}{h} \underset{h \neq 0}{=} h + 2a - 3$.

(c) Use part (b) to find the derivative $f'(a)$.

2 $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} (h + 2a - 3)$

2 $= 2a - 3$

(d) Find the equation of the tangent line to the graph of f at $a = 4$. Put your answer into the form $y = mx + b$. Point $(4, f(4)) = (4, 4)$

1 slope $= f'(4) = 2 \cdot 4 - 3 = 5$. Thus

3 $y - 4 = 5(x - 4)$

$y = 5x - 16$

(b) Slope of secant line = $h + 2a - 3$

(c) Derivative $f'(a) =$ $2a - 3$

(d) Equation is $y = 5x - 16$

(10) (a) State the Intermediate Value Theorem. Use Complete Sentences.

7

Let f be a continuous function on the interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a,b) such that

$$f(c) = N.$$

(b) Find an interval $[a, b]$ of length at most 2 such that the equation

$$4x^5 + 10x^3 + 3x = -1$$

has a solution in (a, b). **Justify your answer using the Intermediate Value Theorem.**

2 Consider $f(x) = 4x^5 + 10x^3 + 3x + 1$.

1 Then f is continuous everywhere and

2 $f(0) = 1$ and $f(-1) = -16$.

1 Choose $N = 0$. Then N is between $f(-1)$ and $f(0)$.

2 Thus, by IVT there exists c in the interval $(-1, 0)$ s.t. $f(c) = 0$.

1 This means

$$4c^5 + 10c^3 + 3c = -1$$

and we have a solution in the interval $(a,b) = (-1, 0)$.