(1) Consider the functions $f(x) = 4^x$ and $g(x) = \frac{1}{x-2}$.

(a) Determine $f(g(1))$ and $g(f(1))$.

(b) Find all numbers a such that $(g \circ f)(a)$ is defined.

(a) $f(g(x)) = f\left(\frac{1}{x-2}\right) = 4^{\frac{1}{x-2}}$, thus $f(g(1)) = 4^{-1} = \underline{\underline{\frac{1}{4}}}$. 3

$g(f(x)) = g(4^x) = \frac{1}{4^x - 2}$, thus $g(f(1)) = \frac{1}{4-2} = \underline{\underline{\frac{1}{2}}}$. 3

(b) $(g \circ f)(x) = g(f(x)) = \frac{1}{4^x - 2}$ is not defined iff $4^x - 2 = 0$

iff $4^x = 2$

iff $x = \frac{1}{2}$.

Hence the domain of $g \circ f$ is $\{a \mid a \neq \frac{1}{2}\} = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$ 2

(a) $f(g(1)) =$ $\frac{1}{4}$, $g(f(1)) =$ $\frac{1}{2}$

(b) $g \circ f$ is defined on: $\{a \mid a \neq \frac{1}{2}\}$

(2) Consider the function $f(x) = \dfrac{4x+1}{3x-2}$. Determine the inverse function of f.

5

Solving $y = f(x) = \dfrac{4x+1}{3x-2}$ for x we get

$$\begin{aligned} y(3x-2) &= 4x+1, \text{ thus} \\ 3xy - 4x &= 1 + 2y, \text{ so} \\ x(3y-4) &= 2y+1. \end{aligned}$$

no deduction
f not
discussed

It follows that $3y-4 \neq 0$ because otherwise $y = \frac{4}{3}$ and the left-hand side of the equation is zero whereas the right-hand side is $2 \cdot \frac{4}{3} + 1 = \frac{11}{3} \neq 0$, a contradiction. Hence we can divide by $3y-4$ and obtain

1

$$x = \frac{2y+1}{3y-4}.$$

Thus the inverse function is $f^{-1}(y) = \dfrac{2y+1}{3y-4}$ or (changing the name of the variable)

1

$$f^{-1}(x) = \frac{2x+1}{3x-4}.$$

$f^{-1}(x) = \dfrac{2x+1}{3x-4}$

(3) Compute the following limits or show that they do not exist:

(a) $\lim_{x\to 4} \frac{2^x - 8}{\sqrt{x}+2}$, (b) $\lim_{h\to 0} \frac{(h+2)^2-4}{h}$, (c) $\lim_{x\to 2} \frac{x+2}{x-2}$.

(a) The function $f(x) = \frac{2^x - 8}{\sqrt{x}+2}$ is continuous (as $\sqrt{x}$ and 2^x are continuous) 1

and $f(4)$ is defined. Hence we get $\lim_{x\to 4} \frac{2^x-8}{\sqrt{x}+2} = f(4) = \frac{2^4-8}{\sqrt{4}+2} = \frac{8}{4} = \underline{\underline{2}}.$ 3

(b) If $h \neq 0$, we have $\frac{(h+2)^2-4}{h} = \frac{h^2+4h+4-4}{h} = \frac{h(h+4)}{h} = h+4.$ 3

Hence we obtain $\lim_{h\to 0} \frac{(h+2)^2-4}{h} = \lim_{h\to 0}(h+4) = 0+4 = \underline{\underline{4}}$ 1 because $h+4$ is

a continuous function.

(c) Since polynomial functions are continuous, we get

3 $\lim_{x\to 2^+} \frac{x+2}{x-2} = \frac{4}{0^+} = \infty$ and

$\lim_{x\to 2^-} \frac{x+2}{x-2} = \frac{4}{0^-} = -\infty.$ Thus, the one-sided limits are different, so

$\lim_{x\to 2} \frac{x+2}{x-2}$ does not exist. 1

(a) $\lim_{x\to 4} \frac{2^x - 8}{\sqrt{x}+2} =$ 2 (b) $\lim_{h\to 0} \frac{(h+2)^2-4}{h} =$ 4 (c) $\lim_{x\to 2} \frac{x+2}{x-2} =$ DNE

(4) Let f be a function such that $\lim_{x\to 2} f(x)$ exists and $\lim_{x\to 2}\left(\frac{1}{x^2-1}f(x)+4x-1\right)=9$.
Use the limit rules to determine $\lim_{x\to 2} f(x)$. Explain your reasoning.

Set $L = \lim_{x\to 2} f(x)$. The limit laws provide

$$\underbrace{\left(\lim_{x\to 2}\frac{1}{x^2-1}\right)}_{=\frac{1}{3}}\cdot\underbrace{\left(\lim_{x\to 2} f(x)\right)}_{=L}+\underbrace{\lim_{x\to 2}(4x-1)}_{=7}=9,$$

3

where the limits are obtained by direct substitution because rational functions are continuous on their domain. 2

Hence, we have the equation $\frac{1}{3}L+7=9$,

so $\frac{1}{3}L = 2$,

thus $\underline{\underline{L = 6}}$. 3

$\lim_{x\to 2} f(x) =$ 6

(5) Let f be a function such that, for all real numbers x near 1,

$$-2^{-x} + 2x + \frac{1}{2} \le f(x) \le x^5 + \log_2 x + 1.$$

Argue that $\lim_{x\to 1} f(x)$ exists and find its value. As usual, justify your answer.

Since exponential, logarithmic, and polynomial functions are continuous, we obtain

2 $\lim_{x\to 1} (-2^{-x} + 2x + \frac{1}{2}) = -2^{-1} + 2\cdot 1 + \frac{1}{2} = 2$ and

2 $\lim_{x\to 1} (x^5 + \log_2 x + 1) = 1^5 + \log_2 1 + 1 = 1 + 0 + 1 = 2$ (using $2^0 = 1$).

2 Hence both limits agree and the Squeeze Theorem gives that $\lim_{x\to 1} f(x)$

2 exists and that its value is 2.

$\lim_{x\to 1} f(x) =$ 2

(6) A particle is moving on a straight line so that its position after t seconds is given by $s(t) = 4t^2 - t$ meters.

(a) Find the average velocity of the particle over the time interval $[1, 2]$.

(b) Determine the average velocity of the particle over the time interval $[2, t]$, where $t > 2$. Simplify your answer.

(c) Find the instantaneous velocity of the particle at time $t = 2$.

(a) The average velocity is $\dfrac{\text{distance travelled}}{\text{time elapsed}} = \dfrac{s(2)-s(1)}{2-1} = \dfrac{(4\cdot 2^2-2)-(4\cdot 1^2-1)}{1}$ 1 1

$= 14 - 3 = \underline{\underline{11}}.$ 1

(b) $\dfrac{s(t)-s(2)}{t-2} = \dfrac{4t^2-t-14}{t-2} = \dfrac{(4t+7)(t-2)}{t-2}$ 2 1 1

$= \underline{\underline{4t+7}}$ if $t \neq 2$. 1

(c) $s'(2) = \lim_{t\to 2} \dfrac{s(t)-s(2)}{t-2}$ 2

1 $= \lim_{t\to 2}(4t+7)$ by (b)

1 $= 4\cdot 2 + 7$ because polynomial functions are continuous

$= \underline{\underline{15}}.$

(a) average velocity over $[1, 2]$: 11 m/s

(b) average velocity over $[2, t]$: $4t+7$ m/s

(c) instantaneous velocity at time $t = 2$: 15 m/s

(7) Using the definition, find the equation of the tangent line to the graph of the function $f(x) = x^2 - 4x$ at $x = 3$. Write your result in the form $y = mx + b$.

The slope of the tangent line is

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{(3+h)^2 - 4(3+h) - (3^2 - 4 \cdot 3)}{h}$$

$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 12 - 4h - 9 + 12}{h}$$

$$= \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} (2+h)$$

$$= 2 \quad \text{by continuity of polynomial functions.}$$

The equation of the tangent line is

$$y - f(3) = f'(3)(x-3), \text{ so}$$

$$y - (-3) = 2(x-3), \text{ thus}$$

$$y = 2x - 9.$$

The equation of the tangent is $y = 2x - 9$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) Define what it means for a function f to be continuous at a. Use complete sentences.

(b) Let c be a number and consider the function

$$f(x) = \begin{cases} c2^x - 3 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$$

Find all numbers c such that the limit $\lim_{x \to 1} f(x)$ exists.

(c) Is there a number c such that the function f in part (b) is continuous at 1? As always, justify your answer.

4

(a) A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$

(b) Since $c2^x - 3$ and $\frac{1}{x} - 2c$ are continuous, we get

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (c2^x - 3) = 2c - 3$ 3

and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (\frac{1}{x} - 2c) = 1 - 2c$ 3

1

The one-sided limits agree iff $2c - 3 = 1 - 2c$ iff $\underbrace{4c = 4 \text{ iff } c = 1}_{2}$.

Hence $\lim_{x \to 1} f(x)$ exists iff $c = 1$

1

(c) If f is continuous at 1, then $\lim_{x \to 1} f(x)$ must exist. By (b), this forces $c = 1$. 1

And if $c = 1$, then $\lim_{x \to 1} f(x) = 2 \cdot 1 - 3 = -1 \neq 1 = f(1)$.

Hence f is never continuous at 1.

} 0

(b) $c =$ 1

(c) yes / no (*circle the correct answer*)

(9) (a) State the Intermediate Value Theorem. Use complete sentences.

(b) Explain why and how you can use this theorem to show that the equation

$$2^x + x^5 + 3x + 1 = 0$$

has a solution strictly between -1 and 0.

1 1 1

(a) If a function f is continuous on the closed interval $[a,b]$ and N is a number strictly between 1 $f(a)$ and $f(b)$, then there is a number 1 c in the open interval (a,b) such that $f(c) = N$.

1 1

(b) Polynomial and exponential functions are continuous on $\mathbb{R}$, thus the function 2 $f(x) = 2^x + x^5 + 3x + 1$ is continuous on $\mathbb{R}$, so in particular on $[-1, 0]$. 2

We have $f(-1) = \frac{1}{2} - 1 - 3 + 1 = -\frac{5}{2} < 0$ 1

and $f(0) = 1 + 0 + 0 + 1 = 2 > 0$. 1

1 1

Hence, applying the IVT with $N=0$, it follows that there is some c in $(0,1)$ with $f(c) = 0$, that is, c is the desired solution.

1

(10) (a) State the definition of the derivative of a function f at a point a. Use complete sentences.

(b) Using the definition, determine the derivative of the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ \dfrac{1}{2x+1} & \text{if } x \geq 0 \end{cases}$$

at 1 and 0 if it exists.

(It is enough to mention one of the two limits.)

✓

3 (a) The derivative of f at a is $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ (or $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$),

1 provided the limit exists.

2 (b) $f'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

1 $= \lim_{h \to 0} \frac{\frac{1}{2(1+h)+1} - \frac{1}{2 \cdot 1 + 1}}{h}$ (h sufficiently close to zero)

3 $= \lim_{h \to 0} \frac{3-(2h+3)}{3 \cdot h \cdot (2h+3)} = \lim_{h \to 0} \frac{-2h}{3h(2h+3)} = \lim_{h \to 0} -\frac{2}{3(2h+3)}$

1 $= \underline{\underline{-\frac{2}{9}}}$ because rational functions are continuous on their domain.

For $f'(0)$ we consider the one-sided limits

2 $\lim_{h \to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h \to 0^-} \frac{1-1}{h} = 0$ and

2 $\lim_{h \to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{2h+1} - 1}{h} = \lim_{h \to 0^+} \frac{1-(2h+1)}{h(2h+1)} = \lim_{h \to 0^+} \frac{-2h}{h(2h+1)}$

$= \lim_{h \to 0^+} -\frac{2}{2h+1} = -2$ by continuity of rational functions.

1 Since the one-sided limits are different, we conclude that $\lim_{h \to 0} \frac{f(h)-f(0)}{h} = f'(0)$ does not exist.

(b) $f'(1) =$ $-\frac{2}{9}$ $\quad f'(0) =$ DNE