MA 113 — Calculus I
Exam 2

Spring 2010
March 9, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Answer Key

Section:

Last four digits of student identification number: ____________

Question	Score	Total
1		10
2		8
3		9
4		12
5		10
6		8
7		10
8		15
9		15
10		15
Free	3	3
		100

(1) Find the derivative of the following functions. You need not simplify your answer.

(a) $g(x) = \sqrt[3]{x^2}\cos(x)$.

$g(x) = x^{2/3} \cdot \cos x$. By product rule

3 $g'(x) = \frac{2}{3} x^{-1/3} \cos x + x^{2/3} \cdot (-\sin x)$

(b) $h(t) = \dfrac{4t^3 - 2t + 1}{7t - 1}$.

By quotient rule

3 $h'(t) = \dfrac{(7t-1)(12t^2-2) - (4t^3-2t+1)\cdot 7}{(7t-1)^2}$

(c) $f(x) = x^2 e^{7x^3 - 3x^2 + 5}$.

Using product rule and chain rule:

4 $f'(x) = 2x e^{7x^3 - 3x^2+5} + x^2 e^{7x^3-3x^2+5} \cdot (21x^2 - 6x)$

(a) $g'(x) =$ $\frac{2}{3} x^{-1/3} \cos x - x^{2/3} \sin x$

(b) $h'(t) =$ $\dfrac{(7t-1)(12t^2-2) - (4t^3-2t+1)\cdot 7}{(7t-1)^2}$

(c) $f'(x) =$ $2x e^{7x^3-3x^2+5} + (21x^4 - 6x^3) e^{7x^3-3x^2+5}$

(2) Given that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, use the limit laws to find the following limits. **Show all your work and give the exact answers.**

(a) $\lim_{t \to 0} \frac{\sin(9t^3)}{2t^3}$.

4

$$= \lim_{t \to 0} \frac{9}{2} \frac{\sin(9t^3)}{9t^3} \underset{x = 9t^3}{\overset{1}{=\!=}} \lim_{x \to 0} \frac{9}{2} \frac{\sin(x)}{x}$$

1

$$\left[\begin{array}{l} t \to 0, \text{ then} \\ x \to 0 \end{array}\right] \leftarrow 1$$

$$= \frac{9}{2} \lim_{x \to 0} \frac{\sin x}{x} = \underline{\underline{\frac{9}{2}}}$$

1

(b) $\lim_{x \to 0} \left(x \cdot \frac{\ln(x + e^2)}{\sin(x)} \right)$.

4

$$= \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \ln(x + e^2)$$

1

$$= \frac{1}{\lim_{x \to 0} \frac{\sin x}{x}} \cdot \lim_{x \to 0} \ln(x + e^2)$$

1

$$= 1 \cdot \ln(e^2) \quad \text{by continuity of } \ln$$

1

$$= 2 \ln(e) = \underline{\underline{2}}$$

1

(a) $\lim_{t \to 0} \frac{\sin(9t^3)}{2t^3} =$ $\frac{9}{2}$

(b) $\lim_{x \to 0} \left(x \cdot \frac{\ln(x + e^2)}{\sin(x)} \right) =$ 2

(3) Consider the function $f(x) = e^{\sin(x)}$ on the domain $[0, 2\pi]$. **Give exact answers!**

(a) Find the equation of the tangent line at $x = \pi$. Write your answer in the form $y = mx + b$.

(2) $f'(x) = e^{\sin x} \cos x$

point $(\pi, f(\pi)) = (\pi, 1)$

(1) slope $f'(\pi) = -1$

(1) $y - 1 = -(x - \pi)$

$\boxed{y = -x + \pi + 1}$

(b) Find all points (a, b), where $0 \leq a \leq 2\pi$, at which the tangent line to the graph of $f(x)$ is horizontal.

Wanted: $f'(x) = 0$

(1) Thus $e^{\sin x} \cdot \cos x = 0$.

(1) This means $\cos x = 0$,

(1) hence $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$

(2) the points are

$(\frac{\pi}{2}, f(\frac{\pi}{2})) = (\frac{\pi}{2}, e)$

$(\frac{3\pi}{2}, f(\frac{3\pi}{2})) = (\frac{3\pi}{2}, e^{-1})$

(a) Equation: $y = -x + \pi + 1$

(b) Point(s): $(\frac{\pi}{2}, e)$, $(\frac{3\pi}{2}, e^{-1})$

(4) A particle is traveling along a straight line. Its position after $t \geq 0$ seconds is given by

$$s(t) = 2t^3 - 18t^2 + 48t + 7 \text{ meters.}$$

(a) Find the velocity $v(t)$ and acceleration $a(t)$ of the particle at time t.

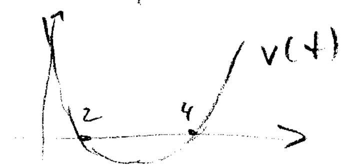
(1) $v(t) = 6t^2 - 36t + 48$ meters/sec

(1) $a(t) = 12t - 36$ meters/sec^2

(b) Find the time interval(s) when the particle is traveling in the positive direction.

(1) $v(t) = 6(t^2 - 6t + 8) = 6(t-4)(t-2)$.

(2) Hence $v(t) > 0$ for t in $[0,2)$ or $(4, \infty)$



(c) Determine the total distance traveled by the particle during the first three seconds.

(1) Particle travels in positive direction for $0 \leq t \leq 2$ and in negative direction for $2 \leq t \leq 3$.

(2) Hence we compute $|s(2) - s(0)| + |s(3) - s(2)|$

(1) $= |16 - 72 + 96 + 7 - 7| + |54 - 162 + 144 + 7 - 16 + 72 - 96 - 7|$

$= 40 + |-4| = \boxed{44 \text{ meters}}$

(d) Find the time interval(s) where the particle is speeding up.

(1) $a(t) = 12t - 36 > 0$ for $t > 3$. Hence

(1) $v(t) > 0$ and $a(t) > 0$ for t in $(4, \infty)$

(1) $v(t) < 0$ and $a(t) < 0$ for t in $(2, 3)$

(a) $v(t) =$ $6t^2 - 36t + 48$ meters/sec, $a(t) =$ $12t - 36$ meters/sec^2

(b) Particle travels in positive direction during $[0,2)$ and $(4,\infty)$

(c) Total distance is 44 meters.

(d) Speeding up during $2 < t < 3$ and $4 < t$.

(5) Consider the curve described given by the equation $(15-4x)\cdot y^2 + 2xy = x^2$. Find the equation of the line tangent to this curve at the point $(3, 1)$. Write your answer in the form $y = mx + b$. **As always, show your work!**

Note: the point is indeed on the curve described by the equation above

Let us write $f(x) = y$. Then

$$(15-4x) f(x)^2 + 2x f(x) = x^2.$$

Differentiating yields

4 $$(15-4x)\cdot 2 f(x) f'(x) - 4 f(x)^2 + 2 f(x) + 2x f'(x) = 2x$$

Thus

2 $$f'(x) = \frac{2x + 4 f(x)^2 - 2 f(x)}{2(15-4x) f(x) + 2x}$$

$$= \frac{x + 2 f(x)^2 - f(x)}{(15-4x) f(x) + x}$$

Substituting 3 for x and 1 for $f(x)$ results in

2 $$f'(3) = \frac{3+2-1}{3\cdot 1+3} = \frac{4}{6} = \frac{2}{3}$$

Point $= (3,1)$, slope $= \frac{2}{3}$

2 $$y - 1 = \frac{2}{3}(x-3) \quad \text{or} \quad \boxed{y = \frac{2}{3}x - 1}$$

Equation of the tangent line is $y = \frac{2}{3}x - 1$

(6) As usual show your work in answering the following questions. **Give the exact answer!**

(a) Find $f'(1)$ if $f(x) = \ln(4x^3 + 7x^2 - 15x + 9)$.

② $f'(x) = \dfrac{12x^2 + 14x - 15}{4x^3 + 7x^2 - 15x + 9}$

① $f'(1) = \dfrac{11}{5}$

(b) Find $g'(\frac{\pi}{2})$ if $g(x) = \sin(x)5^{\cos(x)}$.

③ $g'(x) = \cos x \cdot 5^{\cos x} + \sin x \cdot \ln(5) \cdot 5^{\cos x} \cdot (-\sin x)$

$= (\cos x - \ln(5) \sin^2 x) \cdot 5^{\cos x}$

② $g'(\frac{\pi}{2}) = (0 - \ln(5) \cdot 1) \cdot 5^0 = -\ln(5)$

(a) $f'(1) =$ $\frac{11}{5}$

(b) $g'(\frac{\pi}{2}) =$ $-\ln(5)$

(7) In this problem use the fact that the mass of a radioactive substance changes over time according to the formula $m(t) = m_0 e^{kt}$, where m_0 and k are constants. **Give your answers correct to two decimal places. Show all your work!!**

A radioactive substance known to be 500 years old is 60% decayed (that means, 40% of it is left).

(a) What is the half-life of the substance, this is, after how many years was half of the original amount m_0 decayed?

1 $m(500) = m_0 e^{500k} = \frac{4}{10} m_0 = \frac{2}{5} m_0.$

2 Hence $500k = \ln\left(\frac{2}{5}\right)$ and $\boxed{k = \frac{\ln(\frac{2}{5})}{500}}$

2 Now, $e^{kt} = \frac{1}{2}$ implies $kt = \ln\left(\frac{1}{2}\right)$,

2 hence $t = \frac{\ln(\frac{1}{2})}{k} = \frac{\ln(\frac{1}{2}) \cdot 500}{\ln(\frac{2}{5})} \approx \underline{378.24}$

(b) What percentage of the substance is left after 1000 years?

2 $m(1000) = m_0 e^{1000 \cdot \frac{\ln(\frac{2}{5})}{500}}$

$= m_0 e^{2\ln(\frac{2}{5})} = m_0 e^{\ln(\frac{4}{25})}$

1 $= \frac{4}{25} m_0 = 0.16 m_0$

16 % of m_0 is left after 1000 years.

(a) The half-life is 378.24 years.

(b) After 1000 years there is 16 percent of the original amount left.

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) Consider the function $f(x) = x^2 - 3x + 4$.

(a) Find the equation of the tangent line to the graph of $f(x)$ at the point $(a,\ f(a))$.

2 $f'(x) = 2x - 3$, thus $f'(a) = 2a - 3$

3 $y - f(a) = (2a-3)(x-a)$

$y = (2a-3)(x-a) + a^2 - 3a + 4$

(b) Find all points $(a, f(a))$ on the graph of $f(x)$ such that the tangent line to the graph at $x = a$ passes through the point $(5, 5)$. **As usual, show your work to support your answer.**

2 Plug in 5 for x and y in above equation:

$5 = (2a-3)(5-a) + a^2 - 3a + 4$

$= 10a - 15 - 2a^2 + 3a + a^2 - 3a + 4$,

hence

2 $a^2 - 10a + 16 = 0$

2 $(a-8)(a-2) = 0$ and $a = 8$, $a = 2$.

3 The points are $(8, f(8)) = (8, 44)$ and $(2, f(2)) = (2, 2)$

(a) Equation of the tangent line: $y = (2a-3)(x-a) + a^2 - 3a + 4$

(b) The points are: $(8, 44)$ and $(2, 2)$

(9) Consider the function $f(x) = \cos(x)$ with domain $[0, \pi]$ and let $g(x)$ be the inverse function of $f(x)$.

(a) Give the domain and range of $g(x)$.

2 domain is $[-1, 1]$ (range of $f(x)$)

2 range is $[0, \pi]$ (domain of $f(x)$)

(b) Argue that $\sin(g(x)) = \sqrt{1 - x^2}$. You may use that $\sin^2(x) + \cos^2(x) = 1$.

3 $g(x)$ is in $[0, \pi]$. therefore $\sin(g(x)) \geq 0$

1 and $\sin(g(x)) = \sqrt{1 - \cos^2(g(x))}$

2 $= \sqrt{1 - x^2}$

b/c $\cos(g(x)) = x$.

(c) Use implicit differentiation to show that $g'(x) = \dfrac{-1}{\sqrt{1-x^2}}$. **Show all your work!**

1 We have

$$\cos(g(x)) = x.$$

2 Differentiating yields

$$-\sin(g(x)) \cdot g'(x) = 1$$

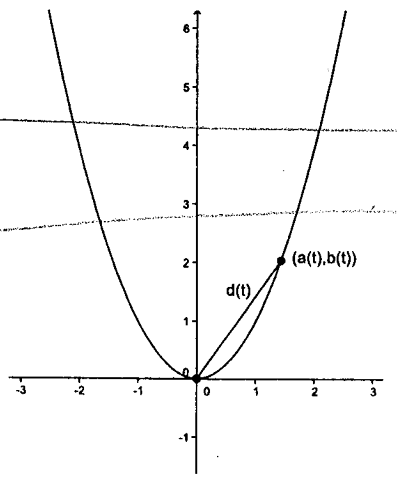
1 Hence

$$g'(x) = -\frac{1}{\sin(g(x))}$$

1

$$\overset{(b)}{=} \frac{-1}{\sqrt{1-x^2}}.$$

(10) A particle is moving along the parabola given by $f(x) = x^2$. At time t seconds its position is denoted by $\big(a(t),\, b(t)\big)$ measured in centimeters, see picture. At the moment when the particle is at the point $(2, 4)$ the first coordinate $a(t)$ increases at rate of 3 cm/sec.



(a) What is the velocity of the second coordinate, $b(t)$, at the moment when the particle is at $(2, 4)$?

2 $b(t) = a(t)^2$. Thus

2 $b'(t) = 2a(t)a'(t)$.

3 Let t_0 be the moment such that $(a(t_0), b(t_0)) = (2,4)$. Then $a'(t_0) = 3$ and therefore

$b'(t_0) = 2 \cdot 2 \cdot 3 = 12$ cm/sec.

(b) At what rate does the distance $d(t)$ of the point $\big(a(t),\, b(t)\big)$ to the origin $(0, 0)$ change at the moment when the particle is at the point $(2, 4)$? **Give the exact answer!**

1 We have

$$d(t)^2 = a(t)^2 + b(t)^2$$

2 Differentiating results in

$$2d(t)d'(t) = 2a(t)a'(t) + 2b(t)b'(t)$$

1 thus

$$d'(t) = \frac{a(t)a'(t) + b(t)b'(t)}{d(t)}$$

4 At time t_0: $a(t_0) = 2$, $b(t_0) = 4$, $a'(t_0) = 3$, $b'(t_0) = 12$.

$$d(t_0) = \sqrt{2^2 + 4^2} = \sqrt{20}.$$

Thus $d'(t_0) = \dfrac{2 \cdot 3 + 4 \cdot 12}{\sqrt{20}} = \dfrac{54}{\sqrt{20}}$ cm/sec.

(a) Velocity of second coordinate is: 12 cm/sec.

(b) Distance changes at a rate of: $\frac{54}{\sqrt{20}}$ cm/sec.