Answer all of questions 1-7 and <u>two</u> out of the three questions 8-10. Please indicate which of questions 8-10 is not to be graded by crossing through its number on the table below. Please turn off cell phones, pages, and any other electronic devices, and please do not wear ear-plugs during the exam. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. **clearly indicate your answer and the reasoning used to arrive at that answer (unsupported answers may not receive credit).**
2. **give exact answers, rather than decimal approximations to the answer**

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Keys

Section: ___________

Last four digits of student identification number: ___________

Question	Score	Total
1		8
2		12
3		10
4		10
5		10
6		10
7		10
8		14
9		14
10		14
Free	2	2
		100

(1) Given that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, use the limit laws to find

(a) $\lim_{t \to 0} \frac{(2t+6)\sin(7t)}{2t}$.

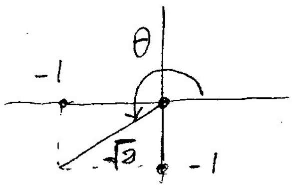
$$\lim_{t \to 0} \frac{(2t+6)\sin 7t}{2t} = \lim_{t \to 0} \left(\frac{7(2t+6)}{2} \cdot \frac{\sin(7t)}{7t} \right) \quad 1pt.$$

$$= \lim_{t \to 0} \frac{7(2t+6)}{2} \cdot \lim_{t \to 0} \frac{\sin(7t)}{7t} \quad 1pt.$$

$$\overset{(u=7t)}{=} \lim_{t \to 0} \frac{7(2t+6)}{2} \cdot \lim_{u \to 0} \frac{\sin u}{u}$$

$$= 21 \cdot 1 = 21 \quad 2pts.$$

(b) If $\cos\theta = \frac{-1}{\sqrt{2}}$ and $\pi < \theta < 3\pi/2$ find the exact value of θ in radians and find $\tan\theta$.
Show your work!



$$\theta = \arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{5\pi}{4} \in \left(\pi, \frac{3\pi}{2}\right) \quad 1pt.$$

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \sin\theta = -\frac{1}{\sqrt{2}} \text{ since for } \pi < \theta < \frac{3\pi}{2},\ \sin\theta < 0 \quad 1pt.$$

$$\Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1 \quad 2pts.$$

(a) limit = 21

(b) θ = $5\pi/4$ $\qquad$ $\tan\theta$ = 1

(2) Use the differentiation rules to find the following derivatives. **Show your work!**

(a) $g'(t)$ at $t = 0$ when $g(t) = \dfrac{3t^2 + t + 8}{2t + 4}$.

(b) $h'(x)$ when $h(x) = \cos(4x) + \cot(x)$.

(c) $f'(x)$ when $f(x) = \dfrac{1}{\sqrt{4 + x^4}}$

(a) $g'(t) = \dfrac{(6t+1)(2t+4) - 2(3t^2+t+8)}{(2t+4)^2}$ 2pts. $\Rightarrow$ $g'(0) = \dfrac{4-16}{16} = -\dfrac{3}{4}$. 2pts

(b) $h'(x) = -\sin(4x)\cdot 4 - \csc^2 x = -4\sin(4x)$ 2pts. $- \csc^2 x$ 2pts.

since $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$.

(c) $f(x) = (4+x^4)^{-1/2}$

$\Rightarrow f'(x) = -\frac{1}{2}(4+x^4)^{-3/2}\cdot(4x^3)$ 3pts. $= -2\left(\dfrac{x}{\sqrt{4+x^4}}\right)^3$. 1pt.

(a) $g'(0) =$ $-3/4$

(b) $h'(x) =$ $-4\sin(4x) - \csc^2 x$

(c) $f'(x) =$ $-2x^3(4+x^4)^{-3/2}$

(3) Use the differentiaion rules to find the following higher order derivatives. **Show your work!**

(a) Find the third derivative of $g(x) = \sin(2x)$. That is find $g^{(3)}(x)$.

(b) Find $h''(0)$ when $h(x) = xe^{3x}$.

(a) $g(x) = \sin(2x)$

$\Rightarrow g'(x) = 2\cos(2x)$ 2pts.

$\Rightarrow g''(x) = -4\sin(2x)$ 2pts.

$\Rightarrow g^{(3)}(x) = -8\cos(2x)$ 1pt.

(b) $h(x) = xe^{3x}$

$\Rightarrow h'(x) = e^{3x} + x \cdot e^{3x} \cdot 3 = (1+3x)e^{3x}$ 2pts.

$\Rightarrow h''(x) = 3 \cdot e^{3x} + (1+3x) \cdot e^{3x} \cdot 3$

$= 3(2+3x)e^{3x}$ 2pts.

$\Rightarrow h''(0) = 6.$ 1pt.

(a) $g^{(3)}(x) =$ $-8\cos(2x)$

(b) $h''(0) =$ 6

(4) A particle is traveling along a straight line. Its position after t seconds is given by $s(t) = 2t^3 - 3t^2 + 50$ meters.

(a) Find the velocity, v, and acceleration, a, of the particle at time t.

$v(t) = s'(t) = 6t^2 - 6t = 6t(t-1)$ m/s — 2pts.

$a(t) = v'(t) = s''(t) = 12t - 6 = 6(2t-1)$ m/s^2 — 2pts.

(b) Determine the time interval (intervals) where the particle is speeding up. Recall that the particle is speeding up at time t if either
(a) $v(t) > 0, a(t) > 0$ or (b) $v(t) < 0, a(t) < 0$.

• $v(t) = 6t(t-1) > 0$ if $t-1 > 0$ & $t > 0$ (the case $t-1<0$ & $t<0$ is ignored since $t \geq 0$) — 1pt

$\Leftrightarrow v(t) > 0$ if $\boxed{t > 1}$ -

$a(t) = 6(2t-1) > 0$ if $2t-1 > 0 \Leftrightarrow \boxed{t > 1/2}$ — 1pt -

$\Rightarrow v(t) > 0$ & $a(t) > 0$ if $\boxed{t > 1}$ — 1pt.

• $v(t) < 0$ if $t-1 < 0$ & $t > 0$ (the case $t-1>0$ & $t<0$ is irrelevant)

$\Leftrightarrow v(t) < 0$ if $\boxed{0 < t < 1}$ — 1pt.

$a(t) < 0$ if $2t-1 < 0 \Leftrightarrow \boxed{0 < t < 1/2}$ — 1pt.

$\Rightarrow v(t) < 0$ & $a(t) < 0$ if $\boxed{0 < t < 1/2}$ - 1pt.

(a) $v(t) =$ $6t(t-1)$ m/s

$a(t) =$ $6(2t-1)$ m/s^2

(b) Time interval (s) $0<t<1/2$ & $t>1$ seconds

(5) As usual **show your work** in answering the following questions!

(a) If $f(x) = \ln\left(\dfrac{x+4}{x-4}\right)$ find $f'(8)$.

$$f'(x) = \frac{1}{\frac{x+4}{x-4}} \cdot \frac{d}{dx}\left(\frac{x+4}{x-4}\right) \quad 2\text{pts.} \quad = \frac{\cancel{x-4}}{x+4} \cdot \frac{x-4-(x+4)}{(x-4)^{\cancel{2}}} = -\frac{8}{(x+4)(x-4)} \quad 2\text{pts.}$$

$$\Rightarrow f'(8) = -\frac{8}{12 \cdot 4} = -\frac{1}{6} \quad 1\text{pt.}$$

Alternatively, $f(x) = \ln(x+4) - \ln(x-4)$ 2pts. $\Rightarrow f'(x) = \dfrac{1}{x+4} - \dfrac{1}{x-4} = -\dfrac{8}{(x+4)(x-4)}$ 2

$$\Rightarrow f'(8) = -\frac{1}{6} \quad 1\text{pt.}$$

(b) If $g(x) = x^4(x+7)^5$ use logarithmic differentiation to find a function $m(x)$ so that $\dfrac{g'(x)}{g(x)} = m(x)$ when $x > 0$.

$$\ln g(x) = \ln\left(x^4(x+7)^5\right) = \ln(x^4) + \ln\left((x+7)^5\right)$$

$$= 4\ln x + 5\ln(x+7) \quad 2\text{pts.}$$

$$\Rightarrow \frac{d}{dx}\left(\ln g(x)\right) = \frac{g'(x)}{g(x)} = \frac{4}{x} + \frac{5}{x+7} \quad 3\text{pts.} \quad = \frac{4(x+7)+5x}{x(x+7)} = \frac{9x+28}{x(x+7)}$$

$$\Rightarrow m(x) = \frac{9x+28}{x(x+7)}$$

(a) $f'(8) =$ $-1/6$

(b) $m(x) =$ $\frac{4}{x} + \frac{5}{x+7} = \frac{9x+28}{x(x+7)}$

(6) Consider the curve given by the equation $xy^3 + 12x^2 + y^2 = 24$. Assume this equation can be used to define y as a function of x (*i.e.*, $y = y(x)$) near (1, 2) with $y(1) = 2$. Find the equation of the tangent line to this curve at (1,2). **Show your work** and write your answer in the form $y = mx + b$.

Differentiating both sides of $xy^3 + 12x^2 + y^2 = 24$ w.r.t. x,

we get : $y^3 + 3xy^2\frac{dy}{dx} + 24x + 2y\frac{dy}{dx} = 0$ 3pts.

$\Rightarrow (3xy^2 + 2y)\frac{dy}{dx} = -24x - y^3$

$\Rightarrow \frac{dy}{dx} = -\frac{24x + y^3}{y(3xy + 2)}$. 2pts.

$\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}} = -\frac{24+8}{2(6+2)} = -\frac{32}{16} = -2 = \boxed{\text{slope of tangent line to the curve } y = f(x) \text{ @ the pt. } (1,2)}$ 2pt

$\Rightarrow$ the eqt. of tangent line @ (1,2) is :

$y - 2 = -2(x-1)$

$\Leftrightarrow y + 2x = 4$. 3pts.

Equation of tangent line $y + 2x = 4$.

(7) The growth rate of the population of a certain colony at time t is given by the differential equation, $P'(t) = kP(t)$, where k is a constant and t is measured in years.

(a) Show that $P(t) = Ae^{kt}$, where A is a constant, is a solution to this differential equation.

$$P(t) = Ae^{kt} \Rightarrow \frac{dP}{dt} = A.e^{kt}.k = kP$$ 2pts.

(b) If intially the colony has 100 members and after two years the colony has 200 members, use algebra to find A and k.

$$P(0) = Ae^{0} = A = 100$$ 2pts. $$\Rightarrow P(t) = 100e^{kt}$$

$$\Rightarrow P(2) = 100e^{2k} = 200$$

$$\Rightarrow e^{2k} = 2 \Rightarrow k = \frac{\ln 2}{2}.$$ 2pts.

(c) Find the rate at which the population is changing with respect to time after 8 years.

Remember to give exact answers!

$$P(8) = 100e^{\frac{\ln 2}{2}\cdot 8} = 100e^{4\ln 2} = 100e^{\ln(2^4)}$$
$$= 100 \cdot 2^4 = 1600$$ 2pts.

$$\Rightarrow \left.\frac{dP}{dt}\right|_{t=8} = kP(8) = \frac{\ln 2}{2}\cdot 1600 = 800\ln 2$$ 2pts

(b) $A =$ 100 $k =$ $\ln 2/2$

(c) rate $=$ $800\ln 2$ members/year

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) A particle moves on the parabola $y = \sqrt{3}\,x^2$ so that its position at time t is given by $(x(t), y(t)) = (x(t), \sqrt{3}x(t)^2)$. As the particle moves through $(1, \sqrt{3})$, $x(t)$ is increasing at the rate of 2 centimeters/second. How fast is the distance from the particle to the origin changing at this instant? Recall that if d is the distance from the origin to the point (x, y), then $d^2 = x^2 + y^2$.

- Let t_0 be the instant @ which $x(t_0) = 1\text{ cm}$, $y(t_0) = \sqrt{3}\text{ cm}$, & $\left.\frac{dx}{dt}\right|_{t_0} = 2\text{ cm/s}$.

- $d^2(t) = x^2(t) + y^2(t) = x^2(t) + 3x^4(t)$ (since $y = \sqrt{3}x^2$) — 4pts.

$\Rightarrow d^2(t_0) = 1 + 3 = 4 \Rightarrow \boxed{d(t_0) = 2\text{ cm}}$, 2pts.

& $2d(t)d'(t) = 2x(t)x'(t) + 12x^3(t)\,x'(t)$ 4pts.

$\Rightarrow d'(t) = \dfrac{x(t)x'(t)}{d(t)}\left(1 + 6x^2(t)\right)$ 2pts.

$\Rightarrow d'(t_0) = \dfrac{1.2}{2}(1+6) = 7\text{ cm/s}$. 2pts.

- Alternatively, $y(t) = \sqrt{3}x^2(t) \Rightarrow y'(t) = 2\sqrt{3}\,x(t)\,x'(t)$ 4pts.

$\Rightarrow y'(t_0) = 2\sqrt{3}.1.2 = 4\sqrt{3}\text{ cm/s}$ 2pts.

$d^2(t) = x^2(t) + y^2(t) \Rightarrow d^2(t_0) = 1+3 = 4 \Rightarrow d(t_0) = 2\text{ cm}$, 2pts

& $2d(t)d'(t) = 2x(t)x'(t) + 2y(t)y'(t)$

$\Rightarrow d'(t) = \dfrac{x(t)}{d(t)}\cdot x'(t) + \dfrac{y(t)}{d(t)}\cdot y'(t)$ 4pts.

$\Rightarrow d'(t_0) = \dfrac{1}{2}\cdot 2 + \dfrac{\sqrt{3}}{2}.4\sqrt{3} = 7\text{ cm/s}$ 2pts.

$d'(t_0) = 7$ cm/s.

(9) (a) State the chain rule. Include all assumptions neccessary to make the rule valid!

If g is differentiable @ x 2pts. & f is differentiable @ $g(x)$, 2pts.

Then $F = f \circ g$ is differentiable @ x &

$F'(x) = f'(g(x)) \cdot g'(x)$. 2pts.

In Leibniz's notation, if $y = f(u)$ & $u = g(x)$ are differentiable,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

(b) Illustrate the chain rule by computing the derivative of $y = \ln(x^2 + x + 1)$. Indicate the functions you are using in the composition.

$f(x) = \ln x$ is diff. for all $x > 0$ & $f'(x) = \frac{1}{x}$. 2pts

$g(x) = x^2 + x + 1$ is diff. for all x in $\mathbb{R}$

& $g'(x) = 2x + 1$. 2pts.

$y = \ln(x^2+x+1) = f(g(x)) = (f \circ g)(x)$.

By the chain rule,

$$\frac{dy}{dx} = \frac{1}{x^2+x+1} \cdot (2x+1) = \frac{2x+1}{x^2+x+1}$$ 4pts.

(b) $\frac{dy}{dx} =$ $(2x+1)/(x^2+x+1)$.

(10) The function $\arctan x$ is defined by $y = \arctan x, -\infty < x < \infty$, if and only if $x = \tan y$, $-\pi/2 < y < \pi/2$. Use implicit differentiation to find the derivative of $\arctan x$. The trigonometric identity $\sec^2 y = 1 + \tan^2 y$ may be helpful in expressing the derivative of $\arctan x$ as a function of x.

- $y = \arctan x$, x in $\mathbb{R}$ $\Leftrightarrow$ $x = \tan y$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

- By implicit differentiation of $x = \tan y$,

$$1 = \sec^2 y \frac{dy}{dx}$$ 4pts.

since $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

$\Rightarrow$ $\boxed{\frac{dy}{dx} = \frac{1}{\sec^2 y}}$. 1pt.

- $\cos^2 y + \sin^2 y = 1$ 2pts.

$\Rightarrow$ $1 + \tan^2 y = \frac{1}{\cos^2 y} = \sec^2 y$ 2pts.

$\Leftrightarrow$ $1 + x^2 = \sec^2 y$ 2pts.

$\Rightarrow$ $\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$. 3pts.

$\frac{d}{dx}(\arctan x) =$ $1/(1+x^2)$.