MA 113 — Calculus I
Exam 2

Fall 2010
October 19, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: ______________________

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		8
2		10
3		10
4		12
5		10
6		10
7		10
8		14
9		14
10		14
Free	2	2
		100

(1) Given that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, use the limit laws to find the following limits. **Show all your work and give the exact answers.**

(a) $\lim_{t \to 0} \frac{\sin(7t^3)}{4t^3}$.

$$= \lim_{t \to 0} \left(\frac{7}{4}\right)\left(\frac{\sin(7t^3)}{7t^3}\right) \underset{x=7t^3}{=\!=} \frac{7}{4} \lim_{x \to 0} \frac{\sin x}{x}$$

1 1

$$= \frac{7}{4} \cdot 1 = \frac{7}{4}$$

1 3

(b) $\lim_{x \to 0} x \ln(x^2 + e^3) \cdot \cot(x)$.

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right) \ln(x^2+e^3)(\cos x)$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right) \cdot \lim_{x \to 0} \ln(x^2+e^3) \cdot \lim_{x \to 0} (\cos x)$$

$$= \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} \cdot \lim_{x \to 0} \ln(x^2+e^3) \cdot \lim_{x \to 0} (\cos x)$$

3

$$= \underbrace{\frac{1}{1} \cdot \ln(0^2+e^3) \cdot \cos 0 = 1 \cdot 3 \cdot 1 = 3}_{2}$$

(a) $\lim_{t \to 0} \frac{\sin(7t^3)}{4t^3} =$ $7/4$

(b) $\lim_{x \to 0} x \ln(x^2 + e^3) \cdot \cot(x) =$ 3

(2) Use the differentiation rules to find the following derivatives. **Show your work. You need not simply your answers.**

(a) $f(x) = \sin(3x) + 2\sqrt{x} + \ln x$.

$$\left.\begin{aligned} f'(x) &= \cos(3x)\cdot 3 + 2\,\tfrac{1}{2}\,x^{-\frac{1}{2}} + \frac{1}{x} \\ &= 3\cos(3x) + x^{-\frac{1}{2}} + \frac{1}{x} \end{aligned}\right\} 3$$

(b) $g(x) = \dfrac{x^2}{x^4+9}$.

$$\underbrace{g'(x) = \frac{2x(x^4+9) - x^2(4x^3)}{(x^4+9)^2} = \frac{18x - 2x^5}{(x^4+9)^2}}_{3}$$

(c) $h(x) = x^3 e^{3x^2+6x+1}$.

$$h'(x) = 3x^2 \underbrace{e^{3x^2+6x+1} + x^3 e^{3x^2+6x+1}}_{3} \frac{d}{dx}(3x^2+6x+1)$$

$$= [3x^2 + \underbrace{x^3(6x+6)] e}_{1}{}^{3x^2+6x+1}$$

(a) $f'(x) =$ $3\cos(3x) + x^{-\frac{1}{2}} + \frac{1}{x}$

(b) $g'(x) =$ $\dfrac{18x - 2x^5}{(x^4+9)^2}$

(c) $h'(x) =$ $[3x^2 + x^3(6x+6)]\, e^{3x^2+6x+1}$

(3) Use the differentiation rules to find the following higher order derivatives. **Show your work.**

(a) Find $f''(x)$ if $f(x) = e^{x^3}$.

$$f'(x) = 3x^2 e^{x^3}$$

2

$$f''(x) = 6x\, e^{x^3} + 3x^2 e^{x^3}(3x^2)$$

$$= [6x + 9x^4]\, e^{x^3}$$

3

(b) Find $g''(1)$ if $g(x) = (x+2) \cdot \sin(\frac{\pi}{4}x)$. **Give the exact answer.**

$$g'(x) = 1 \cdot \sin(\tfrac{\pi}{4}x) + (x+2)\cos(\tfrac{\pi}{4}x) \cdot \tfrac{\pi}{4}$$

2

$$g''(x) = \cos(\tfrac{\pi}{4}x) \cdot \tfrac{\pi}{4} + \cos(\tfrac{\pi}{4}x) \cdot \tfrac{\pi}{4} - \left(\tfrac{\pi}{4}\right)^2 (x+2)\sin(\tfrac{\pi}{4}x)$$

$$= \frac{\pi}{2}\cos(\tfrac{\pi}{4}x) - \left(\tfrac{\pi}{4}\right)^2 (x+2)\sin(\tfrac{\pi}{4}x)$$

2

$$g''(1) = \frac{\pi}{2}\cos(\tfrac{\pi}{4}) - \left(\tfrac{\pi}{4}\right)^2 (1+2)\sin(\tfrac{\pi}{4})$$

$$= \left[\frac{\pi}{2} - 3 \cdot \left(\frac{\pi}{4}\right)^2\right] \frac{\sqrt{2}}{2}$$

1

(a) $f''(x) =$ $(6x + 9x^4)\, e^{x^3}$

(b) $g''(1) =$ $\left[\frac{\pi}{2} - 3\left(\frac{\pi}{4}\right)^2\right] \frac{\sqrt{2}}{2}$

(4) A particle is moving along a straight line. Its position after t seconds is given by $s(t) = t^3 - 3t^2 - 9t + 10$ meters.

(a) Find the velocity, $v(t)$, and the acceleration, $a(t)$, of the particle at time t.

$v(t) = s'(t) = 3t^2 - 6t - 9$ 1

$a(t) = v'(t) = 6t - 6$ 1

(b) Find the time interval(s) when the particle moves in the positive direction.

particle moves in the positive direction if and only if $v(t) > 0$. 1

$v(t) = 3(t^2 - 2t - 3) = 3(t-3)(t+1) > 0$ if and only if $t > 3$. Hence the time interval is $(3, +\infty)$. 2

(c) Determine the total distance traveled by the particle during the first six seconds.

1 The particle moves ~~backward~~ backward during $(0, 3)$ and forward during $(3, 6)$.

2 Total distance $= |s(3) - s(0)| + |s(6) - s(3)|$

1 $= |27 - 27 - 27 + 10 - 10| + |6^3 - 3\cdot 6^2 - 9\cdot 6 + 10 - (27 - 27 - 27 + 10)|$

$= 27 + 81 = 108$.

(d) Find the interval(s) where the particle is speeding up (that means, the time t where $a(t)$ and $v(t)$ have the same sign).

1 $a(t) = 6t - 6 = 6(t-1) > 0$ if and only if $t > 1$

2

	0	1	3
$v(t)$	$-$	$-$	$+$
$a(t)$	$-$	$+$	$+$

speeding up on $[0, 1)$ and $(3, +\infty)$

(a) $v(t) =$ $3t^2 - 6t - 9$ m/s, $a(t) =$ $6t - 6$ m/s^2

(b) Particle travels in positive direction during $(3, +\infty)$

(c) Total distance is 108 meters

(d) Speeding up during $[0, 1)$ and $(3, +\infty)$.

(5) Consider the curve given by the equation $xy^3 + 12x^2 + y^2 = 51$. Assume this equation can be used to define y as a function of x (i.e. $y = f(x)$) near $(2, 1)$. Find the equation of the tangent line to this curve at the point $(2, 1)$. **Show your work** and write your answer in the form $y = mx + b$.

By implicit differentiation, we have :

3 → $y^3 + 3xy^2 \frac{dy}{dx} + 24x + 2y \frac{dy}{dx} = 0$

2 → $\frac{dy}{dx} = -\frac{24x + y^3}{2y + 3xy^2}$

Slope at $(2,1)$ is

2 → $\left.\frac{dy}{dx}\right|_{x=2} = -\frac{24\cdot 2 + 1^3}{2\cdot 1 + 3\cdot 2\cdot 1^2} = -\frac{49}{8}$

2 Equation of the tangent line is:

$y - 1 = -\frac{49}{8}(x-2)$

1 → $y = -\frac{49}{8}x + \frac{53}{4}$

Equation of the tangent line $y = -\frac{49}{8}x + \frac{53}{4}$

(6) As usual, show your work in answering the following questions.

(a) Find $f'(2)$ if $f(x) = \ln\left(\frac{x^2+1}{x^4+1}\right)$

④ $$\begin{cases} f'(x) = \left(\ln(x^2+1)\right)' - \left(\ln(x^4+1)\right)' \\ \quad = \frac{2x}{x^2+1} - \frac{4x^3}{x^4+1} \end{cases}$$

or

$$\begin{cases} f'(x) = \dfrac{\left(\frac{x^2+1}{x^4+1}\right)'}{\left(\frac{x^2+1}{x^4+1}\right)} \\ \quad = \dfrac{2x(x^4+1) - 4x^3(x^2+1)}{(x^2+1)(x^4+1)} \end{cases}$$

① $f'(2) = \frac{2 \cdot 2}{2^2+1} - \frac{4 \cdot 2^3}{2^4+1} = -\frac{92}{85}$

(b) Find $g'(\frac{\pi}{4})$ if $g(x) = \tan(\sin x)$.

③ $$\begin{cases} g'(x) = \sec^2(\sin x) \cdot \frac{d}{dx}(\sin x) \\ \quad = \sec^2(\sin x)\cos x \end{cases}$$

② $$\begin{cases} g'\left(\frac{\pi}{4}\right) = \sec^2\left(\sin\left(\frac{\pi}{4}\right)\right)\cos\left(\frac{\pi}{4}\right) \\ \quad = \frac{\sqrt{2}}{2}\frac{1}{\cos^2\left(\frac{\sqrt{2}}{2}\right)} \end{cases}$$ (or $\frac{\sqrt{2}}{2}\sec^2\left(\frac{\sqrt{2}}{2}\right)$)

(a) $f'(2) =$ $-\,92/165$

(b) $g'(\frac{\pi}{4}) =$ $\frac{\sqrt{2}}{2}\,\frac{1}{\cos^2(\sqrt{2}/2)}$

(7) In this problem use the fact that the mass of a radioactive substance changes over time according to the formula $m(t) = m_0 e^{kt}$, where m_0 and k are constants. **Show all your work and give exact answers.**

A radioactive substance known to be 600 years old is 75% decayed (that means, 25% is left).

(a) What is the half-life of the substance, that is, after how many years was half of the original amount m_0 decayed?

3

$$m_0 e^{k \cdot 600} = 25\% \, m_0 = \frac{1}{4} m_0$$

$$k = \frac{\ln(\frac{1}{4})}{600} = -\frac{\ln 4}{600}$$

3

Set $t_{\text{half-life}} = t^*$. Then $m_0 e^{kt^*} = \frac{1}{2} m_0$

$$\Rightarrow t^* = \frac{\ln \frac{1}{2}}{k} = + \frac{\ln 2}{\frac{\ln 4}{600}} = 300 \text{ yrs.}$$

(b) What percentage of the substance is left after 1200 years?

3

$$m(1200) = m_0 e^{k \cdot 1200} = m_0 e^{-\frac{\ln 4}{600} \times 1200}$$

$$= m_0 e^{-2 \ln 4} = m_0 \cdot \left(\frac{1}{16}\right)$$

1

The percentage is $\frac{m(1200)}{m_0} = \frac{1}{16} = \frac{100}{16}\%$

$= 6.25\%$

(a) The half-life is ______300______ years

(b) After 1200 years there is ____________ percent of m_0 left.

6.25%

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) A particle moves on the curve $y = 2\sqrt{2}x^3$ so that its position after t seconds is given by $P(t) = (x(t), y(t)) = (x(t), 2\sqrt{2}x(t)^3)$, measured in centimeters. As the particle moves through $(1, 2\sqrt{2})$, $x(t)$ is increasing at the rate of 5 centimeters/second. How fast is the distance from the particle to the origin changing at this instant? (Recall that if d is the distance from the point $P = (x, y)$ to the origin, then $d = \sqrt{x^2 + y^2}$.)

The distance square $d^2(t) = x^2(t) + y^2(t)$
$= x(t)^2 + 8x(t)^6$ ③

Differentiating results in

$2d(t)\,d'(t) = 2x(t)\,x'(t) + 8\cdot 6\, x(t)^5 x'(t) \leftarrow$ ④

$d'(t) = \dfrac{x(t) + 24\,x(t)^5}{d(t)}\; x'(t) \leftarrow$ ②

At the point $(1, 2\sqrt{2})$, we have

$x(t) = 1,$

$d(t) = \sqrt{1^2 + 8\cdot 1^6} = \sqrt{9} = 3$ ③

$\frac{dx}{dt} = 5\,cm/s$

Hence $d'(t) = \dfrac{1 + 24\cdot 1^5}{3}\cdot 5 = \dfrac{125}{3}\ cm/s$ ②

$125/3$ cm/s

(9) (a) State the chain rule. Include all the necessary assumptions to make the rule valid.

5

If g is differentiable at x and f is differentiable at $g(x)$ 1, then $f \circ g$ 1 is differentiable at x 1. Moreover

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x) \quad 2$$

(b) Suppose f and g are differentiable functions such that $f(3) = 2$, $f'(3) = 4$, $g(1) = 3$, and $g'(1) = 6$. Find each of the following:
(i) $h'(3)$ where $h(x) = \ln([f(x)]^4)$.
(ii) $l'(1)$ where $l(x) = f(x^4 \cdot g(x))$.

(i) $$h'(x) = \frac{4f(x)^3 f'(x)}{f(x)^4} = 4\frac{f'(x)}{f(x)} \quad 3$$

$$h'(3) = 4 \cdot \frac{f'(3)}{f(3)} = 4 \cdot 4/2 = 8 \quad 1$$

(ii) $$l'(x) = f'(x^4 g(x))\,(x^4 g(x))'$$
$$= f'(x^4 g(x))\,(4x^3 g(x) + x^4 g'(x)) \quad 4$$

$$l'(1) = f'(3)\,(4 \cdot 1^3 \cdot 3 + 1^4 \cdot 6)$$
$$= 4(12+6) = 72 \quad 1$$

(i) 8 (ii) 72

(10) Consider the function $f(x) = \tan(x)$ with domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and let $g(x)$ be the inverse function of $f(x)$.

(a) Give the domain and range of $g(x)$.

2 - Domain of $g = (-\infty, +\infty)$;

2 - Range of $g = (-\frac{\pi}{2}, \frac{\pi}{2})$.

(b) Use implicit differentiation and the formula $\sec^2 y = 1 + \tan^2 y$ to find show that $g'(x) = \frac{1}{1+x^2}$. **You need to show all your work.**

Since $\underbrace{\tan(g(x)) = x}_{2}$, Implicit differeatiation implies

$$\left.\begin{aligned} 1 &= \frac{d}{dx}\left(\tan(g(x))\right) \\ &= \frac{d}{du}(\tan u)\Big|_{u=g(x)} \cdot g'(x) \\ &= \sec^2(g(x)) \cdot g'(x) \end{aligned}\right\} 5$$

$$\left.\begin{aligned} \Rightarrow g'(x) &= \frac{1}{\sec^2(g(x))} = \frac{1}{1+\tan^2(g(x))} \\ &= \frac{1}{1+x^2} \quad (\because \tan(g(x)) = x) \end{aligned}\right\} 3$$