(1) Find the absolute maximum and absolute minimum value of the function $f(x) = 2x^3 - 3x^2 - 12x + 1$ on the interval $[-3, 3]$.

We use the closed interval method. First we find the critical numbers of f. Since

1 $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2)$, we

2 have that $f'(x) = 0$ exactly when $x = -1$ or $x = 2$.

Thus the endpoints $-3$ and $3$ and $-1$ and $2$ are critical numbers.

Next we evaluate f at the critical numbers

| x | f(x) |
|---|---|
| -3 | -44 |
| -1 | 8 |
| 2 | -19 |
| 3 | -8 |

4 $f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 1$
$= -2 \cdot 27 - 27 + 36 + 1 = -44$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1 = 8$

$f(2) = 2(2^3) - 3(2^2) - 12(2) + 1 = -19$

$f(3) = 2(3^3) - 3(3^2) - 12(3) + 1$
$= 2(27) - 27 - 36 + 1 = -8$

Since the absolute maximum and minimum in [3,-3] occur at critical numbers, these are 8 and -44, respectively

1 Absolute maximum value: 8 1 Absolute minimum value: -44

(2) Consider the function $f(x) = x^4 - 4x^3 - 18x^2 + 5x + 1$. Use the concavity test to determine

(a) the interval(s) where the graph of $f$ is concave upward,

(b) the interval(s) where the graph of $f$ is concave downward,

(c) all inflection points of $f$.

If there are none, write NONE.

We use the concavity test.

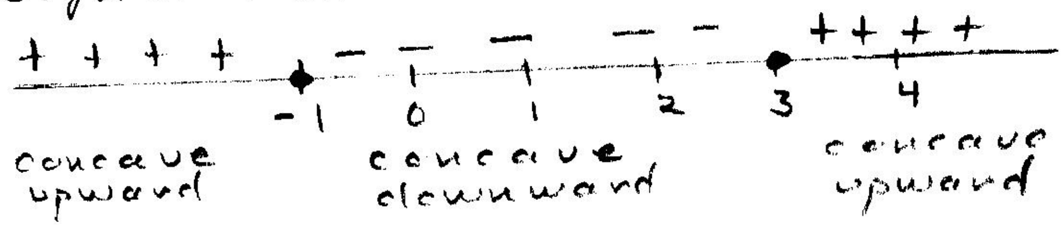
$f'(x) = 4x^3 - 12x^2 - 36x + 5$

2 $f''(x) = 12x^2 - 24x - 36 = 12(x^2 - 2x - 3)$

$= 12(x-3)(x+1)$

sign of $f''(x)$

2




A point of inflection occurs where the graph changes concavity

$f(-1) = (-1)^4 - 4(-1)^3 - 18(-1)^2 + 5(-1) + 1 = -17$

$f(3) = 3^4 - 4(3^3) - 18(3^2) + 5(3) + 1 = -173$

(a) Interval(s) where $f$ is concave upward $(-\infty, -1), (3, \infty)$ 2

(b) Interval(s) where $f$ is concave downward $(-1, 3)$ 1

(c) Point(s) of inflection of $f$ $(-1, -17), (3, -173)$ 2

**(3)** Let $f$ be a function with **derivative** given by $f'(x) = 3(x-2)^2(x-1)^3(x+3)$. Use this derivative to determine

(a) the interval(s) where $f$ is increasing,

(b) the interval(s) where $f$ is decreasing,

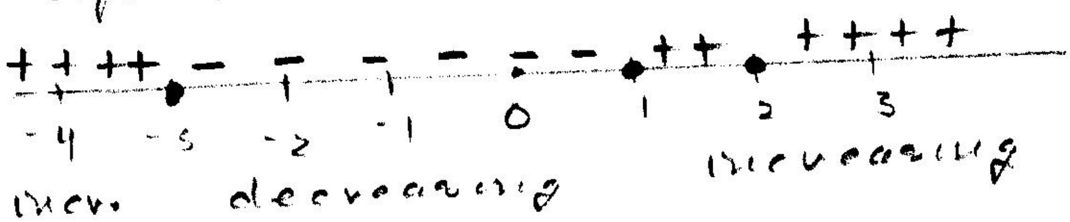
(c) all the values of $x$ where $f$ has a local maximum,

(d) all the values of $x$ where $f$ has a local minimum.

If there are none, write NONE.

We use the first derivative test.

sign of $f'(x)$




(a) Interval(s) where $f$ increasing $(-\infty,-3), (1,\infty)$ 2

(b) Interval(s) where $f$ decreasing $(-3,1)$ 2

(c) Value(s) where $f$ has local maximum $-3$ 2

(d) Value(s) where $f$ has local minimum $1$ 2

**(4)** Use the limit laws to find the following limits:

(a) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9x^8+x}}{2x^4-6}$

(b) $\displaystyle\lim_{x\to\infty} (x-\sqrt{x^2+5})$

a) $$\lim_{x\to\infty} \frac{\sqrt{9x^8+x}}{2x^4-6} = \lim_{x\to\infty} \frac{\sqrt{x^8(9+\frac{1}{x^7})}}{2x^4(1-\frac{3}{x^4})} = \lim_{x\to\infty} \frac{\sqrt{9+\frac{1}{x^7}}}{2(1-\frac{3}{x^4})} \quad 2$$

$$= \frac{\sqrt{9+\lim_{x\to\infty}\frac{1}{x^7}}}{2\left(1-3\lim_{x\to\infty}\frac{1}{x^4}\right)} = \frac{\sqrt{9+0}}{2[1-3(0)]} = \frac{3}{2} \quad 2$$

b) $$\lim_{x\to\infty}(x-\sqrt{x^2+5}) = \lim_{x\to\infty} \frac{(x-\sqrt{x^2+5})(x+\sqrt{x^2+5})}{x+\sqrt{x^2+5}} \quad 2$$

$$= \lim_{x\to\infty} \frac{x^2-(\sqrt{x^2+5})^2}{x+\sqrt{x^2+5}} = \lim_{x\to\infty} \frac{-5}{x+\sqrt{x^2+5}} = 0 \quad 2$$

(a) $\frac{3}{2}$ (b) $0$

(5) Find the most general antiderivative for each of the following functions:

(a) $f(x) = x^5 - 16x^3 + x - 2$

(b) $g(x) = \sin(x) + \cos(x)$

(c) $h(x) = 3\sqrt{x} - \dfrac{1}{x\sqrt{x}}$

a) $F'(x) = x^5 - 16x^3 + x - 2$

$F(x) = \frac{x^6}{6} - 16\left(\frac{x^4}{4}\right) + \frac{x^2}{2} - 2x + C$

$= \frac{x^6}{6} - 4x^4 + \frac{x^2}{2} - 2x + C$

b) $G'(x) = \sin(x) + \cos(x)$

$G(x) = -\cos(x) + \sin(x) + C$

c) $H'(x) = 3\sqrt{x} - \frac{1}{x\sqrt{x}} = 3x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

$H(x) = 3\left(\frac{2x^{\frac{3}{2}}}{3}\right) - \left(-2x^{-\frac{1}{2}}\right) + C$

$= 2x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C$

$= 2\frac{x^2+1}{\sqrt{x}} + C$

(a) General antiderivative of $f$: $\frac{x^6}{6} - 4x^4 + \frac{x^2}{2} - 2x + C$ 3

(b) General antiderivative of $g$: $-\cos(x) + \sin(x) + C$ 3

(c) General antiderivative of $h$: $2x^{3/2} + 2x^{-\frac{1}{2}} + C$ 3

**(6)** A car is traveling along a straight road at 60 ft/s when the driver slams on the brakes in such a way that the acceleration is constant until the car stops. It took the car 5 seconds to stop.

(a) Determine the acceleration while braking.

(b) How far did the car travel after the brakes were applied?

$t$ - number of seconds after braking
$v$ - velocity of the car at time $t$
$a$ - acceleration of the car at time $t$

$$\frac{dv}{dt} = a = c$$

$$v = ct + v_0 \quad (2)$$

When $t = 0$, $v = v_0 = 60$

When $t = 5$, $v = 0$ so

$$0 = c(5) + 60$$
$$5c = -60$$
$$c = -12 \quad (2)$$

$s$ - distance traveled $t$ seconds after braking

$$\frac{ds}{dt} = v = -12t + 60$$

$$s = -12\left(\frac{t^2}{2}\right) + 60t + s_0 \quad (2)$$

when $t = 0$, $s = s_0 = 0$

When $t = 5$,

$$s = -6(5^2) + 60(5) = 30(-5+10) = 150 \text{ ft}$$
(2)

(a) Acceleration $-12$ ft/sec$^2$

(b) Distance the car traveled 150 ft

(7) It is known that a certain function $f$ satisfies $\int_2^5 f(x)\,dx = 7$ and $\int_1^5 f(x)\,dx = 3$. Find the value of:

(a) $\int_1^5 [5 + 3f(x)]\,dx$

(b) $\int_1^2 f(x)\,dx$.

a) $\int_1^5 [5 + 3f(x)]dx = \int_1^5 5dx + 3\int_1^5 f(x)\,dx$ 3

$= 5(4) + 3(3) = 29$ 2

b) $\int_1^2 f(x)dx + \int_2^5 f(x)dx = \int_1^5 f(x)dx$ 2

$\int_1^2 f(x)dx + 7 = 3$

$\therefore \int_1^2 f(x)dx = 3 - 7 = -4$ 2

(a) $\int_1^5 [5 + 3f(x)]\,dx =$ 29

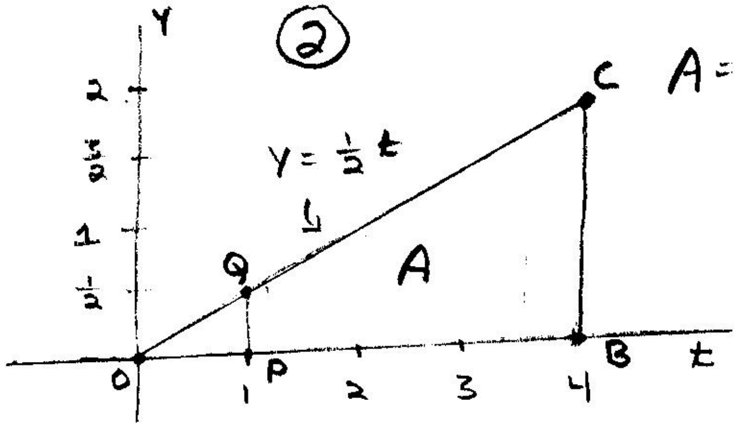
(b) $\int_1^2 f(x)\,dx =$ −4

(8) Interpret each the following integrals as an area. Graph the area and use geometry to find the value of each integral.

(a) $\int_1^4 \frac{1}{2} t \, dt$

(b) $\int_0^3 \sqrt{9 - t^2} \, dt$.

a)



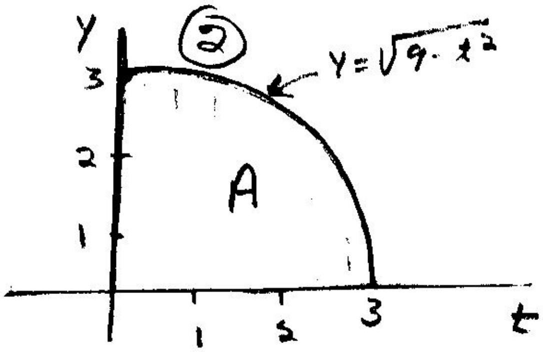

$A = \int_1^4 \frac{1}{2} t \, dt$

$= \text{area}(\triangle OBC) - \text{area}(\triangle OPQ)$

$= \frac{1}{2}(4)(2) - \frac{1}{2}(1)(\frac{1}{2})$

$= 4 - \frac{1}{4} = 3\frac{3}{4}$ 3

b)




$y^2 = 9 - t^2$

$t^2 + y^2 = 9$ circle of radius 3

$A = \frac{\pi 3^2}{4} = \frac{9\pi}{4}$ 2

(a) $\int_1^4 \frac{1}{2} t \, dt =$ $3\frac{3}{4}$

(b) $\int_0^3 \sqrt{9 - t^2} \, dt =$ $9\pi/4$

**Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.**

**(9)** Let $g(x)$ be a function with domain $\{x \mid x \neq 2\}$ that is twice differentiable and satisfies the following conditions:

i) $\lim_{x\to\infty} g(x) = 1, \qquad \lim_{x\to-\infty} g(x) = 0,$

ii) $g(1) = 2, \quad \lim_{x\to 2^+} g(x) = \infty, \quad \lim_{x\to 2^-} g(x) = -\infty,$

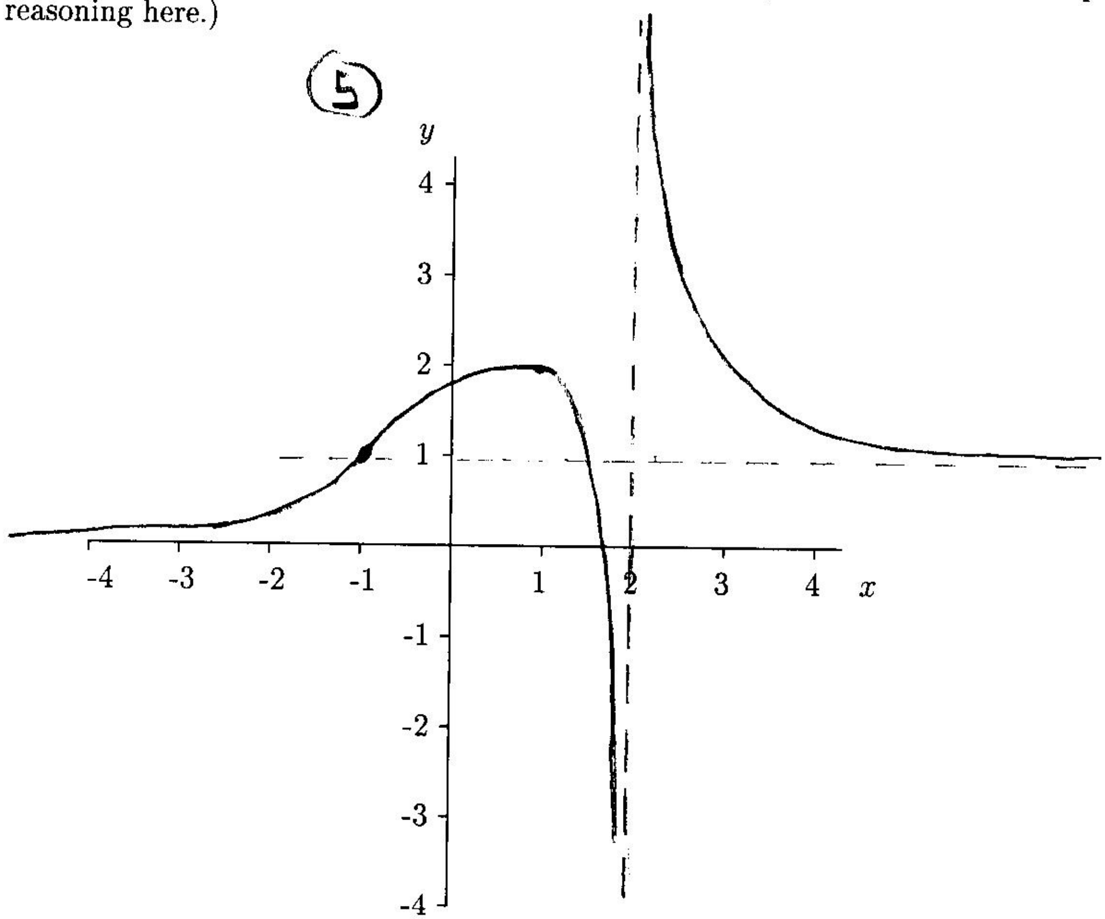
iii) $g'(x) > 0$ when $x < 1$ and $g'(x) < 0$ when $x > 1$ and $x \neq 2$,

iv) $g''(x) > 0$ when $x < -1$ and when $x > 2$,

v) $g''(x) < 0$ when $-1 < x < 2$.

Graph the function $g(x)$ on the axes given below.
Also, provide the information that is asked for below. (You do not need to explain your reasoning here.)

5




Equation(s) of horizontal asymptote(s) $Y = 0 \quad Y = 1$ 2

Equation(s) of vertical asymptote(s) $X = 2$ 2

Critical number(s) 1 2

First coordinate(s) of inflection point(s) $-1$ 2

Absolute maximum value of $f$ NONE 1

**(10)** (a) State the Mean Value Theorem.

If $f$ is continuous on the closed interval $[a,b]$ (2)
and differentiable on the open interval $(a,b)$, (1)
then there exists a number $c$ in $(a,b)$ such (2)
that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. (2)

(b) A train is observed traveling at 30 mi/hr at 1:00 p.m. and at 90 mi/hr at 4:00 p.m.. Use the Mean Value Theorem to show that at some time between these two observations the acceleration of the train is 20 mi/hr$^2$. Your solution should specify the function and the values of the endpoints that you use in the Mean Value Theorem as well as the equation of the Mean Value Theorem.

Let $f(t)$ be the velocity of the train $t$ hours (3)
after noon. Since the instantaneous
acceleration $f'(t)$ can be assumed to exist
when $1 \le t \le 4$, the Mean Value Theorem
applies to $f$ on the interval $[1,4]$. Thus (2)
there is a time $c$ in $(1,4)$ with

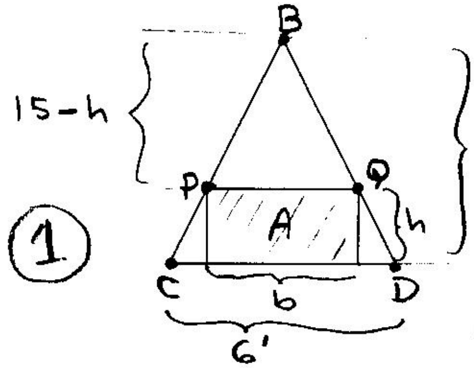
$f'(c) = \dfrac{f(4)-f(1)}{4-1} = \dfrac{90-30}{3} = 20$ mi/hr (2)

**(11)** A rectangle just fits inside an isosceles triangle whose is height is 15 feet and whose base is 6 feet long (as sketched below). Answer the following questions.

(a) Find the height $h$ of the rectangle if $b$ is the length of its base.

(b) Find the height and base of the rectangle of largest area that will fit inside the triangle.

(c) Explain how you know that the area of the rectangle in part (b) is an absolute maximum and not just a local maximum or local minimum.




Since $\triangle BCD$ and $\triangle BPQ$ are similar triangles,
$\frac{15-h}{b} = \frac{15}{6} = \frac{5}{2}$. Then

4

$15-h = \frac{5}{2}b$ so $h = 15 - \frac{5}{2}b$, $0<b<6$.

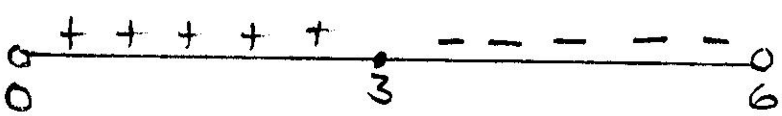
Let $A$ be the area of the rectangle.

1 Then $A = bh = b(15 - \frac{5}{2}b) = 15b - \frac{5}{2}b^2$ so

1 $\frac{dA}{db} = 15 - 5b = 5(3-b)$.

Thus $b=3$ is a critical number.

sign of $\frac{dA}{db}$




2 $A$ has an absolute maximum at $b=3$

2 and $h = 15 - \frac{5}{2}b = 15 - \frac{5}{2}(3) = \frac{15}{2} = 7\frac{1}{2}$

(a) $h$ as a function of $b$: $h = 15 - 5b/2$

(b) $b =$ 3 ft, $h =$ $7\frac{1}{2}$ ft

(c) Reason: Since $\frac{dA}{db} > 0$ on $(0,3)$ and $\frac{dA}{db} < 0$ on

3 $(3,6)$, $A$ is increasing on $(0,3)$ and decreasing on $(3,6)$. Hence $A$ has a maximum on $(0,6)$ at $b=3$ by the first derivative test for absolute extreme values.