MA 113 — Calculus I Fall 2011
Exam 3 November 15, 2011

Answer all of the questions 1 - 8.

Additional sheets are available if necessary. No books or notes may be used. Please turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: ANSWER KEY

Section: __________

Last four digits of student identification number: __________

Question	Score	Total
1		12 = 4 + 4 + 4
2		13 = 7+6
3		14
4		14
5		13
6		13 = 8 + 3+ 2
7		13
8		8
		100

(1) Evaluate the following limits using l'Hôpital's Rule.

(a) $\lim_{x\to -1} \frac{x^5+1}{x+1}$. $= \frac{0}{0}$ Indeterminate form. Use L'Hôpital's rule.

$$\lim_{x\to -1} \frac{5x^4}{1} = 5(-1)^4 = \boxed{5}$$

(b) $\lim_{x\to 0^+} x\ln x$. $= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{+\infty}$ Indeterminate form.

By L'Hôpital's rule

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{-x^2}{1} = \lim_{x\to 0^+} -x = \boxed{0}$$

(c) $\lim_{x\to \pi/2} \frac{\cos^2 x}{1-\sin^3 x}$. $= \frac{0}{0}$ Indeterminate form Use L'Hôpital's rule

$$\lim_{x\to \pi/2} \frac{2\cdot\cancel{\cos x}\cdot(-\sin x)}{-3\sin^2 x\cdot\cancel{\cos x}} \overset{\text{cancel terms}}{=} \lim_{x\to \pi/2} \frac{+2}{3\sin x} = \boxed{\frac{2}{3}}$$

(a) $\lim_{x\to -1} \frac{x^5+1}{x+1} =$ 5

(b) $\lim_{x\to 0^+} x\ln x =$ 0

(c) $\lim_{x\to \pi/2} \frac{\cos^2 x}{1-\sin^3 x} =$ 2/3

(2) (a) Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$.

$f(x) \approx f(a) + f'(a)(x-a)$

So when $a=4$ $\Rightarrow f(4) + f'(4)(x-4)$.

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

$f(4) = 2$ +

$f'(4) = \frac{1}{2\sqrt{4}} = 1/4$

So linearization

$f(x) \approx 2 + \frac{1}{4}(x-4)$

(b) Use the linearization you found in part a to estimate $\sqrt{5}$.

$f(5) \approx 2 + \frac{1}{4}(5-4)$

$= 2\ 1/4$

(a) Linearization of $f(x) = \sqrt{x}$ at $a = 4$ is $2 + \frac{1}{4}(x-4)$

(b) $\sqrt{5} \approx$ $2\ 1/4$

(3) Find the absolute maximum and absolute minimum values of the function

$$f(x) = x^3 - 6x^2 - 15x$$

on the interval $[-2, 3]$.

$$f'(x) = 3x^2 - 12x - 15$$

$f'(x)$ DNE?

Never

(is a polynomial)

$f'(x) = 0$

$3(x^2 - 4x - 5) = 0$

$3(x-5)(x+1) = 0$

$x = 5, \ x = -1$

We're maximizing (+ minimizing) over the interval $[-2,3]$

So just check the values

	x	$f(x)$
endpoints	-2	$(-2)^3 - 6(-2)^2 - 15(-2) = -8 - 24 + 30 = \boxed{-2}$
	3	$3^3 - 6(9) - 45 = 27 - 54 - 45 = \boxed{-71}$
critical pts.	$\cancel{5}$	$\times$ not in interval!
	-1	$-1 - 6(1) - 15(-1) = \boxed{-22}$

absolute max

absolute min.

$$\begin{array}{r} 27 \\ 45 \\ \hline 71 \end{array}$$

The absolute maximum is -2 at $x = -2$.

The absolute minimum is -71 at $x = 3$.

(4) Consider the function $f(x) = xe^x$ on the interval $(-\infty, \infty)$.

(a) Find the interval(s) on which f is increasing and the interval(s) on which f is decreasing.

$$f'(x) = xe^x + e^x \cdot 1 = e^x(x+1)$$

f' DNE

Never

$f'(x) = 0$?

$e^x(x+1) = 0$

e^x never zero

So $x+1 = 0$

$x = -1$

$\searrow \qquad \nearrow$

f' $- - - - - \overset{0}{|} + + + + +$

$f'(-10) \qquad -1 \qquad f'(10) = 10e^{10}$

$f'(-10) = \ominus \oplus = \ominus$

$f'(10) = \oplus$

So decreasing on $(-\infty, -1)$, increasing on $(-1, +\infty)$

(b) Find the interval(s) where the graph of f is concave up or concave down. Show your work.

$$f''(x) = e^x \cdot 1 + (x+1) \cdot e^x$$

$$= (x+2)e^x.$$

Again $f''(x)$ never DNE

concave down, on $(-\infty, -2)$ $\cap$

concave up on $(-2, +\infty)$ $\cup$

$f'' = 0$

$(x+2)e^x = 0$

$x = -2$

f'' $- - \cap - \overset{0}{|} + + + + +$

$f''(-10) = -8 \cdot \oplus = \ominus \oplus = \ominus \qquad -2 \qquad f''(0) = 2 = \oplus$

(c) Find the point(s) of inflection of the graph of f. Show your work.

Since the concavity changes from concave down to concave up at $x = -2$ (see part c), $x = -2$ is the only point of inflection

(a) Interval(s) where f is increasing $(-1, +\infty)$

Interval(s) where f is decreasing $(-\infty, -1)$

(b) Interval(s) where f is concave up $(-2, +\infty)$

Interval(s) where f is concave down $(-\infty, -2)$.

(c) Point(s) of inflection -2.

(5) Find two positive numbers x and y whose product is 49 and whose sum is a minimum.

Minimize $x + y = \overset{s(x)}{\cancel{49}}$ $\quad (x \overset{>}{\cancel{\geq}} 0,\ y \overset{>}{\cancel{\geq}} 0)$

~~Maximize~~ Subject to

$$xy = 49 \Rightarrow y = \frac{49}{x}.$$

So ~~becomes~~

minimize $x + \frac{49}{x} = s(x)$ for $x > 0$,

$$s'(x) = 1 - \frac{49}{x^2}$$

Note $s'(x)$ DNE @ $x=0$, but okay as we have $x>0$

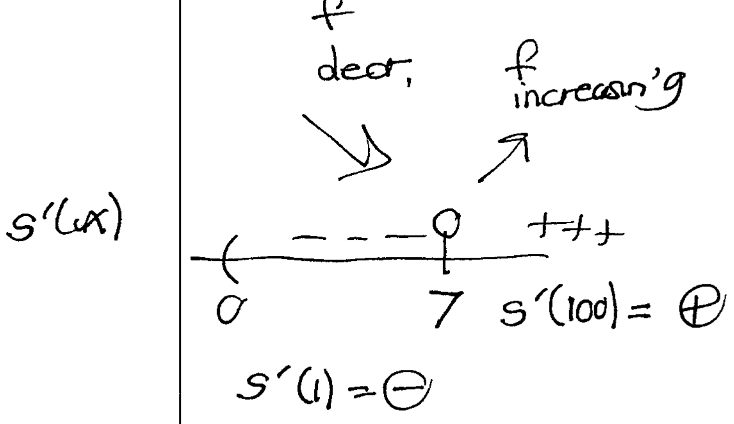
$\underline{s'(x) = 0}$

$$\frac{49}{x^2} = 1$$

$$x^2 = 49$$

$$x = \pm 7.$$

But $x > 0$, so $x = 7$ is the only valid critical point.



So sum is a minimum @ $x = 7$, that is $\overset{x}{\boxed{7}} + \overset{y}{\boxed{\frac{49}{7}}} = 7+7=14.$

$x =$ 7

$y =$ 7

(6) A canister is dropped from a helicopter 490 meters above the ground. The canister has been designed to withstand an impact velocity of at most 100 meters per second.

(a) Given the equation of the acceleration of the canister as a function of time t is

$$a(t) = -9.8 \text{ meters per second squared}$$

(the acceleration is due to gravity), find the velocity $v(t)$ of the canister as a function of time t and the equation of the position $s(t)$ of the canister as a function of time t.

$a(t) = -9.8$

$v(t) = -9.8t + c$ $v(0) = 0$ since canister was just dropped

$\Rightarrow 0 = -9.8(0) + c \Rightarrow c = 0$

$v(t) = -9.8t$

$s(t) = -9.8\frac{t^2}{2} + D.$ But $s(0) = 490 \Rightarrow D = 490$

$\Rightarrow s(t) = -9.8\frac{t^2}{2} + 490,$

(b) At what time is the impact, that is, when does the canister hit the ground?

canister hits ground at t_{splat} ~~after~~ traveling 490 meters, so @ height zero

$s(t_{splat}) = \cancel{490}\, 0 = -9.8\frac{t^2}{2} + 490$

$-490 = -4.9t^2$

$100 = t^2 \Rightarrow t = +10$ sec,

(c) Does the canister survive the impact?

At $t = +10$ sec

$v(10) = -9.8(10) = -98$ meters/sec.

Since 98 m/sec < 100 m/sec

It survives!

(a) $v(t) =$ $-9.8t$

$s(t) =$ $-9.8\, t^2/2 + 490 \quad (= -4.9t^2 + 490)$

(b) Time of impact: $t =$ 10

(c) Does the canister survive the impact? Yes

(7) Doctor Doofenschmirtz is shooting his nemesis Perry the Platypus out of a cannon. He wants Perry the Platypus to land as far away as possible. The distance to the landing point is given by

$$f(\theta) = 450\cos\theta\sin\theta$$

where $0 \leq \theta \leq \pi/2$ is the angle the cannon makes with the ground and the distance is measured in meters. Find the angle θ which maximizes the distance to the landing point.

$$f'(\theta) = 450\cos\theta\cdot\cos\theta + 450\sin\theta(-\sin\theta)$$

$$= 450\,(\cos^2\theta - \sin^2\theta).$$

$f'(\theta) = 0$ when

$\cos^2\theta = \sin^2\theta$

or

$\cos\theta = \pm\sin\theta$.

$f'(\theta)$ DNE?

Never.

For $0 \leq \theta \leq \pi/2$, this is $\theta = \pi/4$.

Maximizing $f(\theta)$ on interval: check endpts & critical.

θ	$f(\theta)$
0	$450\cdot 1 \cdot 0 = 0$
$\pi/2$	$450\cdot 0\cdot 1 = 0$
$\pi/4$	$450\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}} = \frac{450}{2} = 225$ meters.

Max $\to$ $\pi/4$

NOTE: ☆ Easier solution using that

$450\cos\theta\sin\theta$

$\|$

$225(2\sin\theta\cos\theta)$

$\|$

$225\sin 2\theta$

(Easier to differentiate!)

☆

$\theta =$ $\pi/4$

(8) True or False? Circle the correct answers below. For each correct answer, you will score 2 points and for each incorrect answer, you will score 0 points. You do not need to justify your answers.

(a) **TRUE** or [✓ **FALSE.**]
The Mean Value Theorem implies that for a function $f(x)$ such that $f(-1) = 1$ and $f(2) = 2$ there exists a real number c strictly between -1 and 2 such that $f'(c) = 1/3$.

(b) [✓ **TRUE**] or **FALSE.**
Let $f(x) = (e^x - e^{-x})/2$ and $g(x) = (e^x + e^{-x})/2$. The function $f(x)$ is both a derivative and an antiderivative for the function $g(x)$.

(c) **TRUE** or [✓ **FALSE.**]
If $f(x)$ and $g(x)$ are increasing then so is $f(x) - g(x)$.

(d) **TRUE** or [✓ **FALSE.**]
If $f(x)$ and $g(x)$ are differentiable functions with $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$