Exam 3

Form A Solutions

Multiple Choice Questions

1. Let $f(x) = 1/x$. If possible, find the absolute maximum and minimum values for f on the interval $[2, \infty)$.

   A. The maximum is 1/2 and the minimum does not exist.

   B. The maximum is $1/2$ and the minimum is 0.

   C. The maximum is 2 and the minimum is 0.

   D. The maximum is 2 and the minimum does not exist.

   E. The maximum does not exist and the minimum is 0.

2. The volume of a sphere is increasing at a rate of 4π cubic centimeters per minute. Find the rate of change of the radius with respect to time when the radius, r, of the sphere is 3 centimeters. You may use that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

   A. $1/(9\pi)$ centimeters per minute

   B. $1/(4\pi)$ centimeters per minute

   C. $1/(3\pi)$ centimeters per minute

   D. $1/9$ centimeters per minute

   E. $1/4$ centimeters per minute

3. Find the absolute minimum value of $f(x) = |x+1| + 2|x-2|$ on the interval $[-3, 3]$.

   A. -1

   B. 1

   C. 3

   D. 5

   E. 7

4. Suppose f is a differentiable function on the real line, $f(2) = 3$ and $4 \leq f'(x) \leq 6$. Which interval is equal to the set of all possible values for $f(4)$?

   A. $[-5, 15]$

   B. $[7, 9]$

   C. $[10, 12]$

   D. $[11, 13]$

   E. $[11, 15]$

5. On which of the following intervals does the minimum value of the function $f(x) = x^2 - 4x + 3$ occur at an endpoint?

   A. $[0, 2]$

   B. $[0, 3]$

   C. $[0, 4]$

   D. $[1, 3]$

   E. $[1, 4]$

6. Let $f(x) = \dfrac{e^x}{x} - 3$. Which of the statements is true?

   A. f is increasing on $(-\infty, \infty)$

   B. f is increasing on $(-1, 1)$

   C. f is increasing on $(0, 1)$

   D. f is decreasing on $(0, 1)$

   E. f is decreasing on $(1, \infty)$

7. Which condition can guarantee that the function $g(x) = \dfrac{1}{x^2 - 1}$ is concave up?

   A. $x^2 - 1$ is negative

   B. $x^2 - 1$ is positive

   C. $3x^2 - 1$ is negative

   D. $3x^2 - 1$ is positive

   E. $3x^2 + 1$ is positive

8. How many inflection points does the function $h(\theta) = \theta + \cos\theta$ have on the interval $[0, 2\pi]$?

   A. 0

   B. 1

   C. 2

   D. 3

   E. 4

9. Find the value of $\lim_{x \to 0} \dfrac{x^{16} + 11\sin(x)}{2021\ln(1+x)}$.

   A. 11/2021

   B. 16/2021

   C. 27/2021

   D. 11/16

   E. None of the above

10. Evaluate $\lim_{x \to \infty} x \sin(1/x)$.

    A. -1

    B. 0

    C. 1

    D. ∞

    E. None of the above

11. Find the largest area of a rectangle if its perimeter is 20 m.
   A. 10 square meters
   B. 16 square meters
   C. 36 square meters
   D. 25 square meters
   E. 100 square meters

12. Find an antiderivative of $f(x) = 1/x + 11\sin(x) + 16\cos(x)$ on $(0, \infty)$.
   A. $-1/x^2 - 11\cos(x) + 16\sin(x) + C$
   B. $1/x^2 + 11\cos(x) + 16\sin(x) + C$
   C. $\ln(x) + 11\cos(x) + 16\sin(x) + C$
   D. $\ln(x) - 11\cos(x) + 16\sin(x) + C$
   E. $\ln(x) + 11\cos(x) - 16\sin(x) + C$

13. Evaluate the left sum L_4 for $f(x) = 9 - x^2$, $-2 \leq x \leq 2$. In other words find the Riemann sum with four equal-length subintervals, taking the sample points to be the left endpoints.

   A. 26
   B. 30
   C. $30\frac{2}{3}$
   D. 31
   E. 34

14. Use the fact that $\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$ to find

$$\lim_{N\to\infty} \frac{2}{N} \sum_{k=1}^{N} \left(\frac{4k}{N} + 3 \right).$$

   A. 6
   B. 7
   C. 10
   D. 11
   E. 14

15. If $\int_2^8 f(x)\,dx = 17$ and $\int_2^4 f(x)\,dx = 6$, find $\int_4^8 2f(x)\,dx$.

A. -22

B. -11

C. 11

D. 22

E. 46

16. If $\int_1^5 f(x)\,dx = 8$ and $\int_1^5 g(x)\,dx = 3$, find $\int_1^5 (5f(x) - 7g(x) + 2)\,dx$

A. -33

B. 7

C. 13

D. 21

E. 27

Free Response Questions
Show all of your work

17. Merckx road runs north-south and Indurain road runs east-west. Eddy is bicycling south on Merckx road at a speed of 30 kilometers/hour and Miguel is riding east on Indurain road at a speed of 35 kilometers/hour.

    (a) At 12 noon today Eddy is 60 kilometers north the intersection Merckx and Indurain Roads and Miguel is 80 kilometers east of the intersection. What is the distance between these cyclists at 12 noon?

    Solution: Since these measurements indicate that we have a right triangle with sides 60 and 80, the hypotenuse — which is the distance we want — is $z = \sqrt{60^2 + 80^2} = 100$ kilometers.

    (b) What is the rate of change of the distance between the cyclists at 12 noon?

    Solution: Let y be the distance from the intersection to Eddy at time t and let x be the distance from Miguel to the intersection at time t. We are given that Eddy is riding south so that his distance is decreasing and $\frac{dy}{dt} = -30$ kph and Miguel is riding east so that his distance is increasing and $\frac{dx}{dt} = 35$ kph. Setting up the distance between the two bicyclists we have:

$$\begin{aligned} z^2 &= x^2 + y^2 \\ 2z\frac{dz}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{dz}{dt} &= \frac{80(35) + 60(-30)}{100} \\ &= \frac{2800 - 1800}{100} \\ \frac{dz}{dt} &= 10 \text{ kph} \end{aligned}$$

    So, the rate of change of the distance between the cyclists at 12 noon is 10 kilometers/hour.

18. Let $f(x) = x^4 - 8x^2$ for all real numbers x. Using calculus answer the following questions.

(a) At what values of x does f have a local minimum?

Solution: To find the local minima, we have to find the derivative:

$$f'(x) = 4x^3 - 16x$$

set it equal to zero and solve for x to find the critical points.

$$\begin{aligned} 4x^3 - 16x &= 0 \\ 4x(x^2 - 4) &= 0 \\ x &= 0, 2, -2 \end{aligned}$$

We will use the Second Derivative Test to determine which of these are local minima. (Note: You can also use the First Derivative Test.)

$$f''(x) = 12x^2 - 16.$$

$$\begin{aligned} f''(0) &= -16 \text{ so the function is concave down there and this is a local maximum.} \\ f''(-2) &= 32 \text{ so the function is concave up there and this is a local minimum.} \\ f''2 &= 32 \text{ so the function is concave up there and this is a local minimum.} \end{aligned}$$

Thus, the function has local minima at $x = -2$ and at $x = 2$.

(b) What are the x-coordinate(s) of the inflection point(s)?

Solution: The inflection points are where the function changes concavity. We can find the possible inflection points by solving $f''(x) = 0$ for x.

$$12x^2 - 16 = 03x^2 - 4 \qquad = 0x = \pm\frac{2}{\sqrt{3}}$$

We need to check that the second derivative changes sign at each of these two points. If $x < -\frac{2}{\sqrt{3}}$, then the second derivative is positive. If $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$, the second derivative is negative. If $x > \frac{2}{\sqrt{3}}$ then the second derivative is again positive. Thus, the second derivative changes sign at these two points, so the function changes concavity at these two points and the x-coordinates of the inflection points are $x = -2/\sqrt{3}$ and $x = 2/\sqrt{3}$.

(c) On what interval(s) is the function concave up?

Solution: In part (b) we checked the signs of the second derivative and found that the function is concave up on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and on $\left(\frac{2}{\sqrt{3}}, \infty\right)$.

19. Using calculus, find the point on the curve $y = \sqrt{2x}$ that is closest to the point $(3, 0)$ for $x \geq 0$.

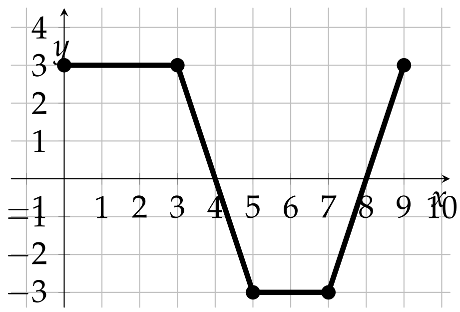
Solution: The distance from a point on the curve $(x, y) = (x, \sqrt{2x})$ to the point $(3, 0)$ is given by the distance formula:

$$\begin{aligned} z &= \sqrt{(x-3)^2 + (y-0)^2} \\ &= \sqrt{(x-3)^2 + (\sqrt{2x})^2} \\ &= \sqrt{x^2 - 6x + 9 + 2x} \\ &= \sqrt{x^2 - 4x + 9} = (x^2 - 4x + 9)^{1/2} \end{aligned}$$

We need to find the derivative, set it equal to zero and solve for x to find the critical points.

$$\frac{dz}{dx} = \frac{1}{2}(x^2 - 4x + 9)^{-1/2}(2x - 4) = \frac{x - 2}{\sqrt{x^2 - 4x + 9}}.$$

Setting this equal to 0, we find that $x = 2$ is our only critical point. We do have one endpoint at $x = 0$. We need to check that we have a global minimum at our critical point. We will use the First Derivative Test. If $0 < x < 2$ then $z'(x) < 0$ and the function is decreasing. If $x > 2$ then $z'(x) >)$ so the function is increasing. Therefore, the function has a global minimum at $x = 2$. Thus, the point on the curve $(x, \sqrt{2x})$ is the point $(2, \sqrt{2 \times 2}) = (2, 2)$.



20. The graph of f is shown above. Evaluate each integral by interpreting it in terms of areas.

(a) $\int_3^0 f(x)\,dx$

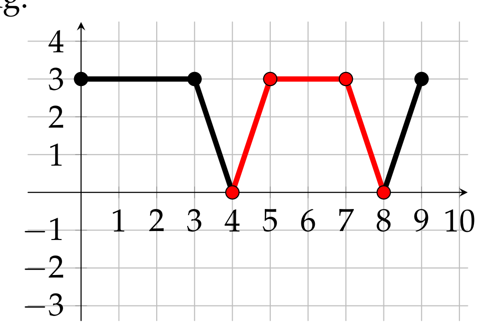
Solution: Since we are going from right to left in the integral — from a greater number to a lesser number — the integral is the negative of the area under the curve. That region is a 3×3 square, so the integral is $\int_3^0 f(x)\,dx = -9$.

(b) $\int_5^7 f(x)\,dx$

Solution: Since the region lies under the x-axis, the integral will be the negative of the area of the region. The region is a rectangle of height 3 and width 2, so area 6, thus $\int_5^7 f(x)\,dx = -6$.

(c) $\int_5^9 |f(x)|\,dx$

Solution: Here we are integrating the absolute value of the function. It's graph is the following:



The integral from 5 to 9 consists of the rectangle from 5 to 7 of height 3 and width 2, the triangle from 7 to 8 of height 3 and width 1, and the triangle from

8 to 9 of height 3 and width 1. The integral will be the sum of those areas:

$$\int_5^9 |f(x)|\,dx = 3(2) + \tfrac{1}{2}(1)(3) + \tfrac{1}{2}(1)(3) = 9.$$