Calculus I
Exam 3

Common exam
11 April 2006

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. When answering these questions, please be sure to 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*.
Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name Key

Section

Last four digits of student identification number Cox

Question	Score	Total
p. 1		14
p. 2		14
p. 3		14
p. 4		14
p. 5		14
Q10		14
Q11		14
Q12		14
Free	2	2
		100

1. Find the absolute maximum and absolute minimum values of $f(x) = x^2 + x$ on the interval $[-1, 2]$.

$f'(x) = 2x + 1$

The only critical value is when $f'(x) = 0$ or $X = -\frac{1}{2}$

X	$f(x)$
-1	0
2	6
$-\frac{1}{2}$	$-\frac{1}{4}$

Thus the minimum is $-\frac{1}{4}$ and occurs when $X = -\frac{1}{2}$ while the maximum is 6 for $X = 2$.

Absolute minimum value $-\frac{1}{4}$, Absolute maximum value 6

2. Suppose that f is a function defined for all real numbers and *the derivative* of f is $f'(x) = x^4$. Find all critical numbers of f. Determine if f has a local maximum, local minimum or no local extremum at each critical point. Justify your answer.

The only critical values is when $f'(x) = 0$ or $X = 0$.
Since $f'(x) > 0$ for all $X \neq 0$, f is an increasing function and so 0 is neither a local maximum nor a local minimum.

Critical number(s) 0

Neither

3. Find the following limits. Show your work. (One may not determine a limit by computing a few values.)

(a) $\lim_{x\to\infty} \frac{2x^2+x}{3x^2-1000x}$

(b) $\lim_{x\to\infty} (\sqrt{x^2+x}-x)$

(a) $\lim_{x\to\infty} \frac{2x^2+x}{3x^2-1000x} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{3-\frac{1000}{x}}$

$= \frac{2+0}{3-0} = 2/3$

(b) $\lim_{x\to\infty} \sqrt{x^2+x}-x = \lim_{x\to\infty} \sqrt{x^2+x}-x \, \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$

$= \lim_{x\to\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1}$

$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

(a) $2/3$, (b) $1/2$

4. Consider the function $f(x) = x^4 - 2x^3 - 12x^2 + 2$. Find the intervals where f is concave up or concave down and determine the inflection points for f.

$f'(x) = 4x^3 - 6x^2 - 24x$

$f''(x) = 12x^2 - 12x - 24 = 12(x^2-x-2) = 12(x-2)(x+1)$

	$x<-1$	$-1<x<2$	$2<x$
Sign $f'(x)$	+	−	+
	up	Down	up

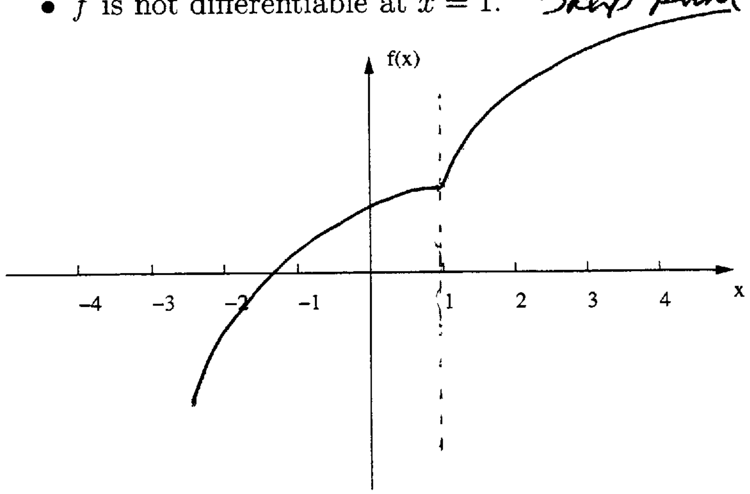
Inflection point(s) at $x =$ -1 and 2

Interval(s) where f is concave up $(-\infty,-1)$ and $(2,\infty)$

Interval(s) where f is concave down $(-1,2)$

5. On the axes below, sketch the graph of a function f which satisfies:

- f is continuous for all x
- $f'(x) > 0$ and $f''(x) < 0$ for x in the interval $(1, \infty)$, Increasing and down
- $f'(x) > 0$ and $f''(x) < 0$ for x in the interval $(-\infty, 1)$ Increasing and down
- f is not differentiable at $x = 1$. Sharp point



6. Find the function $p(x)$ which satisfies $p''(x) = \sin(x)$ and $p(0) = \pi$ and $p(\pi) = 2\pi$.

$p'(x) = -\cos(x) + C$

$p(x) = -\sin(x) + Cx + D$

$\pi = p(0) = -\sin(0) + C \cdot 0 + D$ so $D = \pi$

$2\pi = p(\pi) = -\sin(\pi) + C \cdot \pi + \pi$ so $\pi = \pi C$ and $C = 1$

$p(x) =$ $-\sin(x) + x + \pi$

7. The function f defined by $f(x) = x^4 + 3x^2 + 2x$ has one critical number.

(a) Write an equation whose solution is the critical number for f.

(b) Carry out one step of Newton's method to solve the equation you wrote in part (a). Let $x_0 = 1$ and find the value of x_1. Show how you computed the value x_1.

$f'(x) = 4x^3+6x+2$

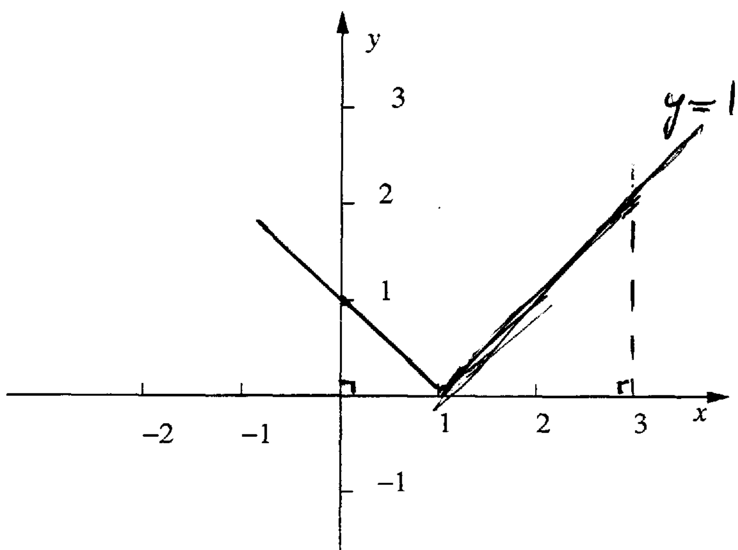
Only critical number is when $f'(x)=0$ or $4x^3+6x+2=0$

$$x_{n+1} = x_n - \frac{4x_n^3+6x_n+2}{12x_n^2+6x_n}$$ Thus, if $x_0 = 1$, we have

$$x_1 = 1 - \frac{4(1)^3+6(1)+2}{12(1)^2+6(1)} = 1 - \frac{12}{18} = 1 - 2/3 = 1/3$$

(a) $4x^3+6x+2=0$, (b) $x_1 =$ $1/3$

8. Sketch the graph of $f(x) = |x-1|$ and use a geometric argument to compute the integral $\int_0^3 |x-1|\,dx$.



$\int_0^3 |x-1|dx$ = area bound by $y=|x-1|$, $x=0$, $x=3$ and the x-axis.

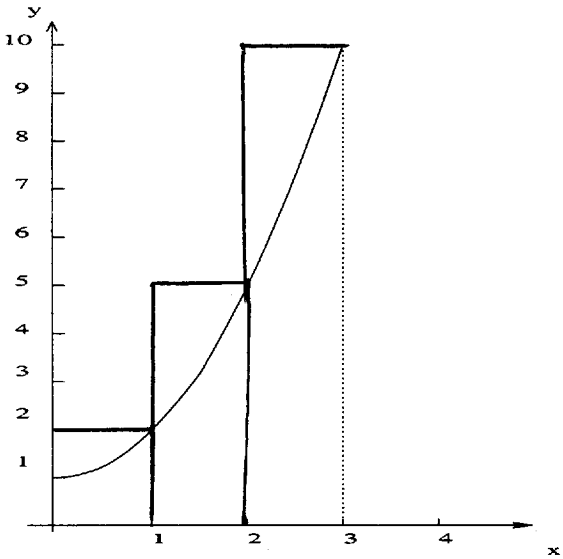
This is the area of two right triangles whose areas are $\frac{1}{2}(1)(1)$ and $\frac{1}{2}(2)(2)$ which equals $\frac{1}{2}+2$ = ~~4.5~~ $=2.5$

$\int_0^3 |x-1|\,dx =$ ~~4.5~~ 2.5

9. Compute the value of the Riemann sum for $\int_0^3 (x^2+1)\,dx$ which is obtained by dividing the interval $[0,3]$ into 3 equal subintervals and using the right endpoint of each subinterval as the sample point.

On the graph below, sketch the rectangles whose area we compute to find the value of the Riemann sum.

Is the value of the Riemann sum smaller or larger than the value of the integral?



$\Delta x = \frac{3-0}{3} = 1$

$x_0 = 0 \quad x_0^* = 1$

$x_1 = 1 \quad x_2^* = 2$

$x_2 = 2 \quad x_3^* = 3$

$x_3 = 3$

Riemann Sum $= (1^2+1)(1) + (2^2+1)(1) + (3^2+1)(1)$

$= 2 + 5 + 10 = 17$

Value of Riemann sum 17

The value of the Riemann sum is LARGER than the value of the integral.

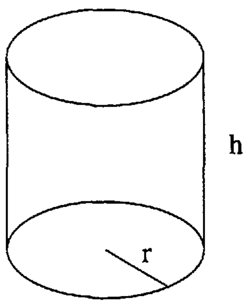
Answer two of the following three questions. Indicate the question that is not to be graded by marking through this question on the front of the exam.

10. A can to be made in the shape of a right-circular cylinder. The volume of the can is to be 1000 centimeters^3. The can does not have a top.

(a) Express the surface area of the sides and bottom of the can as a function of r, the radius of the can.

(b) Find the radius and height of the can type described in part (a) which has least surface area.

(c) Explain how you know you have found the absolute minimum.



$$V = \pi R^2 H = 1000 \text{ so } H = \frac{1000}{\pi R^2}$$

a) $$S = 2\pi R H + \pi R^2 = 2\pi R\left(\frac{1000}{\pi R^2}\right) + \pi R^2$$
$$S = \frac{2000}{R} + \pi R^2$$

b) $$\frac{dS}{dR} = -\frac{2000}{R^2} + 2\pi R \text{ so } \frac{dS}{dR} = 0 \text{ when}$$
$$\frac{2000}{R^2} = 2\pi R \text{ or } R^3 = \frac{1000}{\pi} \text{ or } R = \frac{10}{\sqrt[3]{\pi}}$$

In this case $$H = \frac{1000}{\pi R^2} = \frac{1000}{\pi \frac{100}{\pi^{2/3}}} = \frac{10}{\sqrt[3]{\pi}}$$

so $H = R$.

c) $$\frac{d^2S}{dR^2} = \frac{4000}{R^3} + 2\pi \text{ and } \left.\frac{d^2S}{dR^2}\right|_{R=\frac{10}{\sqrt[3]{\pi}}} > 0$$

So by the Second Derivative Test $\frac{10}{\sqrt[3]{\pi}}$ is a local minimum. Because there's only one positive critical number this local minimum is also an absolute minimum.

• Or $A'(r) < 0$ for $r < 10/\sqrt[3]{\pi}$

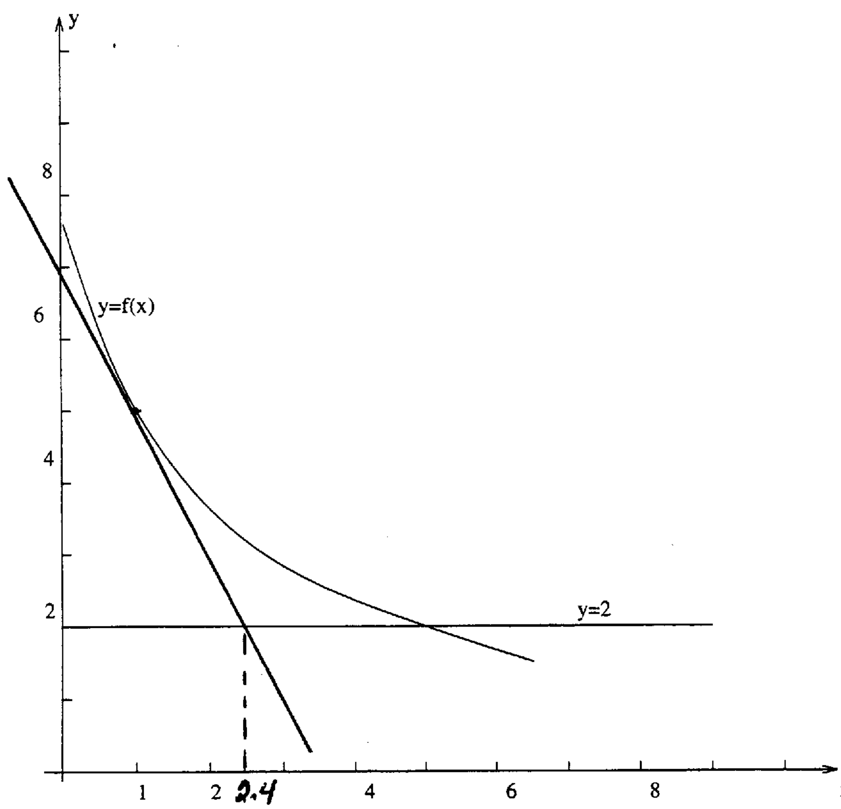
$A'(r) > 0$ for $r > 10/\sqrt[3]{\pi}$ so $10/\sqrt[3]{\pi}$ is a minimum

11. In this problem, you are asked to derive a variant of Newton's method for solving an equation of the form $f(x) = 2$.

(a) Write the linear approximation to the function f at the point a.

(b) Solve the equation $L(x) = 2$ for x. Give your answer in terms of a, $f(a)$ and $f'(a)$.

(c) Consider the function whose graph appears below. Carry out the procedure described in parts (a) and (b). Sketch the tangent line to the graph of f at $a = 1$ and estimate the x-coordinate of the point where this tangent line intersects the line $y = 2$.



Point $= P(a, f(a))$

Slope $= f'(a)$

(a) $L(x) = f(a) + f'(a)(x-a)$

(b) $2 = L(x) = f(a) + f'(a)(x-a) = f(a) + f'(a)x - af'(a)$

Thus $x = \dfrac{2 - f(a) + af'(a)}{f'(a)} = a - \dfrac{f(a)}{f'(a)} + \dfrac{2}{f'(a)}$

c) $x \approx 2.4$

12. Let $f(x) = \frac{1}{4-x^2}$.

(a) Find the y-intercept(s) of the graph of f.

(b) Find all horizontal and vertical asymptotes to the graph of f.

(c) Compute $f'(x)$ and give the domain of $f'(x)$.

(d) Find all critical numbers of f.

(e) Use the first derivative to determine the intervals of increase and decrease for f and find all local extrema for f.

(f) Compute $f''(x)$ and give the domain of $f''(x)$.

(g) Use the second derivative to find intervals of concavity for f.

(h) Sketch the graph of f and label all local extrema. (Use the axes on the next page.)

a) Y intercept is $(0, \frac{1}{4})$. Since $f(x) \neq 0$ there is no x-intercept

b) $\lim_{x \to \infty} \frac{1}{4-x^2} = 0$ so x axis is a horizontal asymptote

$\lim_{x \to \pm 2} \frac{1}{4-x^2} = \pm\infty$ so lines $X = 2$ and $X = -2$ are vertical asymptotes

c) $f'(x) = \frac{2x}{(4-x^2)^2}$. Domain is all reals except ± 2.

d) Only critical number is when $f'(x) = 0$ or $X = 0$

e)

Sign $f'(x)$	$X < 0$	$0 < X$
	$-$	$+$
	Dec	Inc

Increase $(0, 2)$, $(2, \infty)$

Decrease $(-\infty, -2)$, $(-2, 0)$

$X = 0$ is a local minimum

f) $f''(x) = \frac{(4-x^2)(2) - (2x)\,2(4-x^2)(-2x)}{(4-x^2)^4} = \frac{8+6x^2}{(4-x^2)^3}$

Domain of $f''(x)$ is all reals except ± 2

g)

Sign $f''(x)$	$X < -2$	$-2 < X < 2$	$2 < X$
	$-$	$+$	$-$
	Down	up	Down

Concave up $(-2, 2)$

Concave Down $(-\infty, -2)$, $(2, \infty)$

7)