Name: ______________________ Section and/or TA: ______________

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems. You do not need to compute a decimal approximation to your answer. For example, the answer 4π is preferred to 12.57.

Multiple Choice Questions

1	(A)	(B)	(C)	(D)	(E)	**7**	(A)	(B)	(C)	(D)	(E)
2	(A)	(B)	(C)	(D)	(E)	**8**	(A)	(B)	(C)	(D)	(E)
3	(A)	(B)	(C)	(D)	(E)	**9**	(A)	(B)	(C)	(D)	(E)
4	(A)	(B)	(C)	(D)	(E)	**10**	(A)	(B)	(C)	(D)	(E)
5	(A)	(B)	(C)	(D)	(E)	**11**	(A)	(B)	(C)	(D)	(E)
6	(A)	(B)	(C)	(D)	(E)	**12**	(A)	(B)	(C)	(D)	(E)

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) A 14 foot ladder is leaning against a wall. If the top slips down the wall at a rate of 3 ft/s, how fast will the foot be moving away from the wall when the top is 11 feet above the ground?

   A. 0.35 feet per second

   B. 4.21 feet per second

   C. 5.1 feet per second

   D. 3.81 feet per second

   E. 1.85 feet per second

2. (5 points) Suppose a population at time t consists of $P(t) = 100e^{0.0113t}$ individuals. Find the time it takes for the population to double.

   A. 100

   B. $\ln(0.0113)/2$

   C. 113

   D. 200

   E. $\ln(2)/0.0113$

3. (5 points) Let $f(x) = \ln(1 + x)$. Find the quadratic approximation to f at $x = 0$.

   A. $1 - x - \frac{1}{2}x^2$

   B. $1 + x + \frac{1}{2}x^2$

   C. $1 - x - \frac{x^2}{2}$

   D. $x + \frac{1}{2}x^2$

   E. $x - \frac{1}{2}x^2$

4. (5 points) Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x^2 - 9x + 5$ on the interval $[0, 4]$.

   A. absolute maximum 0; absolute minimum -22

   B. absolute maximum 5; absolute minimum -22

   C. absolute maximum 5; absolute minimum -15

   D. absolute maximum does not exist; absolute minimum -15

   E. absolute maximum 5; absolute minimum does not exist

5. (5 points) Let $f(x) = 1/x$. If possible, find the absolute maximum and minimum values for f on the interval $[2, \infty)$.

   A. The maximum does not exist and the minimum is 0.

   B. The maximum is 2 and the minimum does not exist.

   C. The maximum is 1/2 and the minimum does not exist.

   D. The maximum is 1/2 and the minimum is 0.

   E. The maximum is 2 and the minimum is 0.

6. (5 points) Find the intervals where the function

$$f(x) = x^3 - 3x + 1$$

is increasing and the ones where it is decreasing.

   A. is always decreasing

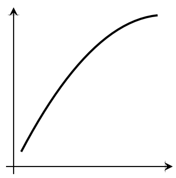
   B. is always increasing

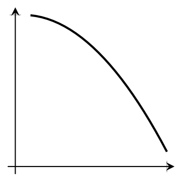
   C. increasing on $(-\infty, -3) \cup (3, \infty)$ and and decreasing on $(-3, 3)$

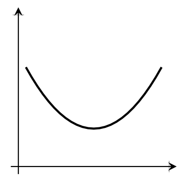
   D. increasing on $(-1, 1)$ and decreasing on $(-\infty, -1) \cup (1, \infty)$

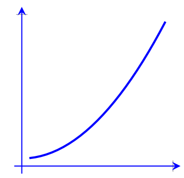
   E. increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$

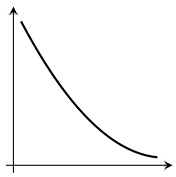
7. (5 points) Select the graph of the function which is increasing and concave up.

A. 

B. 

C. 

D. 

E. 

8. (5 points) Let $f(x) = \ln(2x^2 + 2)$. You are given that

$$f'(x) = \frac{4x}{2x^2 + 2} \quad \text{and} \quad f''(x) = \frac{8 - 8x^2}{(2x^2 + 2)^2}$$

On what intervals is f concave up or down?

A. f is concave up on $(1, +\infty)$ and concave down on $(-\infty, 1)$

B. f is concave up on $(-1, 1)$ and concave down on $(-\infty, -1) \cup (1, +\infty)$

C. f is concave up on $(-\infty, 0)$ and concave down on $(0, +\infty)$

D. f is concave up on $(0, +\infty)$ and concave down on $(-\infty, 0)$

E. f is concave up on $(-\infty, -1) \cup (1, +\infty)$ and concave down on $(-1, 1)$

9. (5 points) Find the value of $\lim_{x \to 0} \frac{x^2 + 113\sin(x)}{2023\ln(1+x)}$

A. 0

B. $\frac{2}{2023}$

C. $\frac{226}{2023}$

D. $\frac{113}{2023}$

E. Does not exist

10. (5 points) Use L'Hôpital's rule to evaluate the following limit

$$\lim_{x \to 0^+} 113\sqrt{x} \cdot \ln x$$

A. 0

B. 1

C. -1

D. 113

E. The limit does not exist

11. (5 points) Find the general antiderivative of $f(x) = 2 + \cos(x) + \dfrac{1}{\sqrt{1-x^2}}$.

A. $-\sin(x) + \dfrac{-2x}{1-x^2} + C$

B. $2x + \sin(x) + \ln(\sqrt{1-x^2}) + C$

C. $2x + \arctan(x) + C$

D. $\dfrac{\cos(x)^2}{2} + \ln(\sqrt{1-x^2}) + C$

E. $2x + \sin(x) + \arcsin(x) + C$

12. (5 points) Find F, an antiderivative of $f(x) = \sqrt{x} + 13\sin(x) + 15\cos(x)$ on $(0, \infty)$, with $F(0) = 100$.

A. $\dfrac{1}{2\sqrt{x}} + 13\cos(x) - 15\sin(x) + 115$

B. $\dfrac{2}{3}x^{3/2} - 13\cos(x) + 15\sin(x) - 113$

C. $\dfrac{2}{3}x^{3/2} + 13\cos(x) - 15\sin(x) + 115$

D. $\dfrac{2}{3}x^{3/2} - 13\cos(x) + 15\sin(x) + 113$

E. $\dfrac{1}{2\sqrt{x}} + 13\cos(x) - 15\sin(x) + 113$

Free response questions: Show work clearly with proper notation.

13. (10 points) The half-life of carbon-14 is 5730 years. This means that its mass will decrease to half its original value after 5730 years. The function $M(t) = M_0 e^{-kt}$ gives the mass of carbon-14 after t years.

    (a) Find the decay rate k.

    (b) A particular piece of wood contains about 37% as much carbon-14 as plants do today. Estimate the age of the piece of wood. **Hint**, set $M(T) = 37\% M(0)$ and find T.

Solution: Compare WS 3.3 # 3, Review WS #2.

a) We want to find k so that $e^{-k5730} = 1/2$.

Applying the function ln, we have $-k5730 = \ln(1/2) = -\ln(2)$ and then $k = \ln(2)/5730$. Using a calculator $k \approx 1.20968 \times 10^{-4} = 0.000120968\,\text{years}^{-1}$.

b) We need $e^{-kT} = 37/100$ and applying the natural logarithm, we obtain the equation $-kT = \ln(0.37)$. Solving for T, gives $T = -\ln(0.37)/k$. As a decimal $T \approx 8219$ years.

Grading. Exponential equation for k, $e^{-k5730} = 1/2$. (3 points). Result of applying ln (1 point). Value for k (1 point). Accept either $\ln(2)/5730$ or $-\ln(1/2)/5730$.

Units are not expected for k, but give a gold ⋆ to students who have that right.

b) Equation for T, $e^{-kT} = 37/100$ (1 point). Apply ln correctly (1 point). Solve for T (1 point) and substitute value for k (1 point). Units for T (1 point).

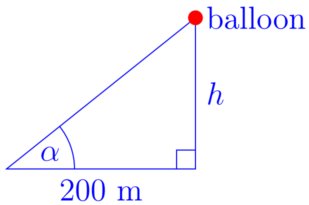
14. (10 points) A helium balloon rising vertically is tracked by an observer located 200 meters from the lift-off point.

(a) Make a sketch summarizing the information in the problem. Label the quantities you use in your solution.

(b) At a certain moment, the *angle* between the observer's line-of-sight and the horizontal is $\frac{\pi}{4}$, and it is changing at a constant rate of 0.05 radians per second. How fast is the balloon rising at this moment?

Solution:

a)



b) We have $\tan\alpha = h/200$. Thus, $h = 200\tan\alpha$. Then

$$\frac{dh}{dt} = 200\sec^2\alpha\frac{d\alpha}{dt}$$

but $\frac{d\alpha}{dt} = 0.05$ when $\alpha = \pi/4$ and $\sec(\pi/4) = \sqrt{2}$. Thus,

$$\frac{dh}{dt} = 200\left(\sqrt{2}\right)^2(0.05) = 20$$

Thus, the balloon is rising 20 meters per second.

Grading:

a) Sketch with at least one side or angle labeled (3 points)

b) Equation involving α and h (2 points). Differentiate to find equation involving $\frac{dh}{dt}$ (2 points). Find $\sec(\pi/4)$ (1 point). Find answer for $\frac{dh}{dt}$ (1 point). Units in part a) and b) (1 point).

15. (10 points) Let $f(x) = e^{-\frac{(x-1)^2}{2}}$. Clearly use calculus to answer the following:

(a) Find the intervals over which $f(x)$ is increasing, and find the intervals over which $f(x)$ is decreasing.

(b) Find each value x where f has a local maximum and each value x where f has a local minimum. Explain how you determine if each is a local maximum or minimum.

(c) Find the intervals over which $f(x)$ is concave up, and find the intervals over which $f(x)$ is concave down.

(d) Find all inflection points.

Solution: a)-b) $f'(x) = (1-x)e^{-(x-1)^2/2}$.

Signs of $f'(x)$:

		1	
$f'(x)$	++		- -

- f is increasing on $(-\infty, 1)$.
- f is decreasing on $(1, \infty)$.
- There is a local maximum at $x = 1$. But since this is the only maximum, it is a global maximum.

c)-d)

$f''(x) = x(x-2)e^{-(x-1)^2/2}$. Then $f''(x) = 0$ implies that $x(x-2) = 0$.

Solutions of the last equation: $c_1 = 0$ and $c_2 = 2$.

		0		2	
x	- -		++		++
$x-2$	- -		- -		++
$f''(x)$	++		- -		++

- f is concave up on $(-\infty, 0)$ and $(2, \infty)$.
- f is concave down on $(0, 2)$.
- $c_1 = 0$ and $c_2 = 2$ are inflection points.

Grading: a) $f'(x)$ (2 points), intervals of increase and decrease (1 point), use sign of f' to decide on increase or decrease (1 point).

Accept open intervals $(-\infty, 1)$ and $(1, \infty)$ as intervals for increase or decrease.

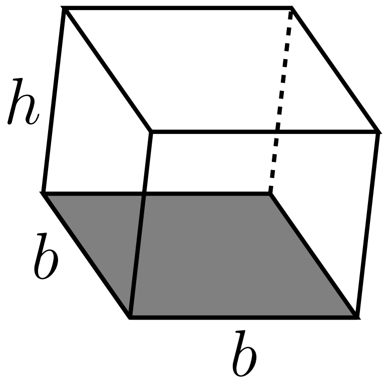
b) Local max. at $x = 1$ (1 point), reason for global max (1 point).

c) Compute $f''(x)$ (1 point). Give intervals of concavity (1 point), use sign of f'' to characterize intervals (1 point)

d) List inflection point (1 point). (Do not deduct if y-coordinate omitted.)

16. (10 points) Clearly show steps using calculus in your solution.

Find the dimensions of an open rectangular box (that is, no top) with a square base that holds 1000 cubic cm and is constructed with the least building material possible.



The volume of the box is $V = b^2h$ and the surface area is $S = 4bh + b^2$.

Solution: We want to minimize $S = 4bh + b^2$ with the relation $V = b^2h = 1000$.

We solve for h in the last equation to find $h = \dfrac{1000}{b^2}$. Substitute into the equation for surface to find S as $S(b) = \frac{4000}{b} + b^2$.

Thus to find the minimum surface, we want to find the minimum value of S for $b > 0$.

We compute $S'(b) = -\frac{4000}{b^2} + 2b = \frac{2b^3-4000}{b^2}$ and find that the critical with $b > 0$ is $b = \sqrt[3]{2000} = 12.5992$. For this value of b, $h = 6.29961$.

This will be a minimum since $S'(b) < 0$ for $0 < b < \sqrt[3]{2000}$ and $V'(b) > 0$ for $\sqrt[3]{2000} < b$.

The box with smallest surface will have $b = \sqrt[3]{2000} = 12.5992$ cm and $h = 6.29961$ cm.

Grading: Solve area for h (2 points), substitute to express S in terms of b (2 points), find critical point (3 points), give b and h for maximum volume (2 points), units on final answer (1 point).

Problem does not ask for justification that we have found a minimum.