MA 113 — Calculus I
Exam 3

Fall 2010
November 16, 2010

Answer all of the questions 1 - 7 and **two** of the questions 8 - 10. Please indicate which problem is not to be graded by crossing through its number in the table below.

Additional sheets are available if necessary. No books or notes may be used. Please, turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities. Please:

1. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
2. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

Name: Key

Section: ____________

Last four digits of student identification number: ____________

Question	Score	Total
1		10
2		9
3		9
4		8
5		12
6		12
7		8
8		14
9		14
10		14
Free	4	4
		100

(1) Consider the function $g(t) = t^4 + 8t^3 + 18t^2 + 5$ on the interval $(-\infty, \infty)$.

(a) Find the critical number(s) of $g(t)$.

(+1) g is differentiable everywhere, so critical numbers occur where $g'(t)=0$,

(+1) $g'(t) = 4t^3 + 24t^2 + 36t$
$= 4t(t^2+6t+9) = 4t(t+3)^2$,

(+1) Critical numbers: $t=0$, $t=-3$,

(b) Find the interval(s) of increase and decrease for $g(t)$.

	Interval	$4t$	$(t+3)^2$	g'	g
(+1)	$(-\infty, -3)$	$-$	$+$	$-$	decreasing
(+1)	$(-3, 0)$	$-$	$+$	$-$	decreasing
(+1)	$(0, \infty)$	$+$	$+$	$+$	increasing

(c) Find all local extreme values of $g(t)$. For each extremum, give both coordinates and specify whether it is a local minimum or a local maximum.

(+2) At the critical value $t=-3$, g' does not change sign (it is negative on both sides). By the First Derivative Test there is no extreme value at $t=-3$.

(+2) At $t=0$, g' changes from negative to positive so by the first derivative test there is a local minimum there, $g(0)=5$.

(a) Critical number(s): $t=0$, $t=-3$

(b) Interval(s) of increase and decrease: Decreasing: $(-\infty,-3)$ and $(-3,0)$
Increasing: $(0,\infty)$

(c) Local maxima (both coordinates): None
Local minima (both coordinates): $g(0)=5$, at $t=0$.

(2) Let $f(x) = (x+4)^{1/3}$.

(a) Find the linear approximation $L(x)$ to $f(x)$ at $x = 4$.

(+1) $f(4) = (4+4)^{1/3} = 8^{1/3} = 2.$

(+2) $f'(x) = \frac{1}{3}(x+4)^{-2/3}\frac{d}{dx}(x+4) = \frac{1}{3}(x+4)^{-2/3}$

$f'(4) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}\cdot\frac{1}{8^{2/3}} = \frac{1}{12},$

(+2) $L(x) = f(4) + f'(4)(x-4)$

$= 2 + \frac{1}{12}(x-4).$

(b) Use the linear approximation you found in part (a) to estimate $(8.25)^{1/3}$. **Present your answer as a rational number.**

(+2) First notice that $8.25 = 4.25 + 4$, so

$(8.25)^{1/3} = f(4.25).$

(+2) But $f(4.25) \approx L(4.25)$

$= 2 + \frac{1}{12}(4.25 - 4)$

$= 2 + \frac{1}{12}\cdot\frac{1}{4} =$

$= 2\frac{1}{48}.$

(a) $L(x) =$ $2 + \frac{1}{12}(x-4)$

(b) $(8.25)^{1/3} \approx$ $2\frac{1}{48}$

(3) Consider the function

$$h(x) = x + \frac{1}{x}.$$

Find the absolute minimum and maximum values of h on the interval $\left[\frac{1}{2}, 2\right]$. Be sure to specify all the values of x where the absolute minimum and maximum are achieved.

We use the Closed Interval Test. (+2)

Critical Values:

$h'(x) = 1 - \frac{1}{x^2} = 1 - x^{-2}$.

h is always differentiable on $(\frac{1}{2}, 2)$.

$h'(x) = 0$ when $1 - x^{-2} = 0$, or when $x = \pm 1$. The only critical value in $[\frac{1}{2}, 2]$ is thus $x = 1$. (+2)

$h(1) = 1 + \frac{1}{1} = 2$. (+1)

Endpoints:

$h(\frac{1}{2}) = \frac{1}{2} + (\frac{1}{2})^{-1} = \frac{1}{2} + 2 = \frac{5}{2}$.

$h(2) = 2 + \frac{1}{2} = \frac{5}{2}$. (+1)

Compare.

Absolute minimum: $h(1) = 2$ at $x = 1$

Absolute maximum: $h(\frac{1}{2}) = h(2) = 5/2$, at $x = \frac{1}{2}$ and $x = 2$. (+3)

The absolute maximum $5/2$ is taken on at $x = \frac{1}{2}$ and $x = 2$.

The absolute minimum 2 is taken on at $x = 1$.

(4) Consider the function f defined on the interval $(0, \pi)$ by $f(\theta) = \frac{\theta^2}{4} + \sin\theta$.

(a) Find the interval(s) where the graph of f is concave up or concave down; show your work.

$f'(\theta) = \frac{2\theta}{4} + \cos\theta = \frac{\theta}{2} + \cos\theta$

$f''(\theta) = \frac{1}{2} - \sin\theta.$ (+1)

$f''(\theta) = 0$ when $\frac{1}{2} - \sin\theta = 0$, or $\sin\theta = \frac{1}{2}$,

In the interval $(0,\pi)$, $\sin\theta = \frac{1}{2}$ at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, (+1)

Also, $\sin\theta < \frac{1}{2}$ on $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$, so $f''(\theta) = \frac{1}{2} - \sin\theta$ is positive on $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$,

$\sin\theta > \frac{1}{2}$ on $(\frac{\pi}{6}, \frac{5\pi}{6})$, so $\frac{1}{2} - \sin\theta < 0$ on this interval, (+2)

Concave up: $f''(\theta) > 0$ on $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$

Concave down: $f''(\theta) < 0$ on $(\frac{\pi}{6}, \frac{5\pi}{6})$. (+2)

(b) Find the point(s) of inflection of the graph of f; show your work.

$f''(\theta) = 0$ at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

The concavity changes at both $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, so both are inflection points. (+2)

$f(\frac{\pi}{6}) = \frac{(\frac{\pi}{6})^2}{4} + \sin\frac{\pi}{6} = \frac{\pi^2}{144} + \frac{1}{2}$

$f(\frac{5\pi}{6}) = \frac{(\frac{5\pi}{6})^2}{4} + \sin\frac{5\pi}{6} = \frac{25\pi^2}{144} + \frac{1}{2}$

(a) Interval(s) where the graph is concave up: $(0, \frac{\pi}{6})$ and $(\frac{5\pi}{6}, \pi)$

Interval(s) where the graph is concave down: $(\frac{\pi}{6}, \frac{5\pi}{6})$

(b) Point(s) of inflection: $(\frac{\pi}{6}, \frac{\pi^2}{144} + \frac{1}{2})$ and $(\frac{5\pi}{6}, \frac{25\pi^2}{144} + \frac{1}{2})$,

(5) Find the general antiderivative each of the following functions.

(a) $f(x) = \frac{1}{4}x^3 + 6x^2 + 1$.

$$\int\left(\tfrac{1}{4}x^3+6x^2+1\right)dx = \tfrac{1}{4}\cdot\tfrac{x^4}{4}+6\cdot\tfrac{x^3}{3}+x+C$$
$$= \tfrac{1}{16}x^4+2x^3+x+C.$$

+4

(b) $g(t) = \csc^2 t - \sin t$.

$$\int(\csc^2 t - \sin t)\,dt = -\cot t + \cos t + C.$$

+4

(c) $h(v) = \frac{1}{v^2} + e^v$.

$$h(v) = v^{-2} + e^v.$$
$$\int(v^{-2}+e^v)\,dv = \tfrac{v^{-1}}{-1} + e^v + C = -\tfrac{1}{v} + e^v + C$$

+4

(a) $F(x) = \frac{1}{16}x^4 + 2x^3 + x + C$

(b) $G(t) = -\cot t + \cos t + C$

(c) $H(v) = -\frac{1}{v} + e^v + C$

(6) Consider the function

$$f(x) = \frac{\sqrt{x^6+2}+1}{x^3-8}.$$

Find all vertical and horizontal asymptotes of the curve $y = f(x)$. Be sure to compute all limits that are needed to justify your answer.

Horizontal asymptotes:

(+3)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(\sqrt{x^6+2}+1)}{(x^3-8)} \cdot \frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}}\sqrt{x^6+2} + \frac{1}{x^3}}{1-8/x^3} = \lim_{x\to\infty} \frac{\sqrt{1+2/x^6} + \frac{1}{x^3}}{1-8/x^3}$$

$$= \frac{\sqrt{1+2\lim_{x\to\infty} 1/x^6} + \lim_{x\to\infty} \frac{1}{x^3}}{1-8\lim_{x\to\infty} 1/x^3} = \frac{\sqrt{1+0}+0}{1-0} = 1.$$

When $x<0$, $\frac{1}{x^3}$ is negative, and $\frac{1}{x^3} = -\sqrt{\frac{1}{x^6}}$. Thus

(+3)

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{\frac{1}{x^3}(\sqrt{x^6+2}+1)}{\frac{1}{x^3}(x^3-8)} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^6}(x^6+2)} + \frac{1}{x^3}}{1-8/x^3}$$

$$= \frac{-\sqrt{1+2\lim_{x\to-\infty} 1/x^6} + \lim_{x\to-\infty} 1/x^3}{1-8\lim_{x\to-\infty} 1/x^3} = -1.$$

(+2) Horizontal asymptotes: $y=1$ and $y=-1$.

Vertical Asymptotes:

(+3)

$$\lim_{x\to2^-} f(x) = -\infty$$

$$\lim_{x\to2^+} f(x) = +\infty,$$

$\longrightarrow$ Vertical asymptote is $x=2$. (+2)

Horizontal asymptotes: $y=1$ and $y=-1$.

Vertical asymptotes: $x=2$.

(7) Evaluate the following limits using l'Hopital's Rule:

(a) $\lim_{x\to\pi} \frac{\cos x + 1}{x^2 - \pi^2}$.

+1 $\cos \pi = -1$, so this is a $\frac{0}{0}$-type indeterminant form.

+3 $\lim_{x\to\pi} \frac{\cos x + 1}{x^2 - \pi^2} \overset{L}{=} \lim_{x\to\pi} \frac{-\sin x}{2x}$

$= \frac{-\sin \pi}{2\pi} = -0/2\pi = 0.$

(direct substitution)

(b) $\lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right)$.

+1 $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, so this is an $\infty \cdot 0$ indeterminant form.

+1 Writing $x(\arctan x - \frac{\pi}{2}) = \frac{\arctan x - \frac{\pi}{2}}{1/x}$, we obtain a $\frac{0}{0}$-form.

+2 $\lim_{x\to\infty} x(\arctan x - \frac{\pi}{2}) = \lim_{x\to\infty} \frac{\arctan x - \frac{\pi}{2}}{1/x} \overset{L}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^2}}{-1/x^2}$

$= \lim_{x\to\infty} \frac{-x^2 \left(\frac{1}{x^2}\right)}{1+x^2 \left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{-1}{1+\frac{1}{x^2}} = -1.$

(a) $\lim_{x\to\pi} \frac{\cos x + 1}{x^2 - \pi^2} =$ 0

(b) $\lim_{x\to\infty} x\left(\arctan x - \frac{\pi}{2}\right) =$ -1.

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(8) (a) State Fermat's Theorem. Use complete sentences.

+1 (If f has a local maximum or minimum at c, and if $f'(c)$ +1

exists, then $f'(c)=0$.

+1

Parts (b) and (c): State whether the given functions have any local extreme values on the given intervals. If so, use an appropriate test to determine whether the extrema are local minima or local maxima. If not, explain how you can be sure that there are no extreme values.

(b) $f(x) = 0.99x + \cos x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

+1 ($f'(x) = .99 - \sin x$.

+2 (Critical value: $f'(x)=0$ when $\sin x = .99$. The only such value of x in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is $x = \arcsin(0.99)$.

+2 ($f''(x) = -\cos x$, which is negative on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. By the Second Derivative Test, f thus has a local maximum at $x = \arcsin(0.99)$.

(c) $g(x) = 1.01x + \cos x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

+1 ($g'(x) = 1.01 - \sin x$.

Setting $g'(x)=0$, we see that $g'(x)=0$ requires that $\sin x = 1.01$.

+2 (Since $\sin x$ is always between -1 and 1, there are no values of x for which $g'(x)=0$.

Also, $g'(x)$ always exists. Thus g has no critical values.) +1

+2 (By Fermat's Theorem, g cannot have any local extreme values, since if it did, g would also have to have critical values.

(9) (a) State the Mean Value Theorem. Use complete sentences.

Assume the function f is:

(+2) 1) continuous on the closed interval $[a,b]$, and
2) differentiable on the open interval (a,b).

Then there is a number c in (a,b) where

(+2) $$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Let $f(x) = (x-4)^2$. Without finding c, use the Mean Value Theorem to show that there is a number c in the interval $(3,5)$ such that $f'(c) = 0$.

(+2) f is continuous on $[3,5]$ and differentiable on $(3,5)$,

(+1) By the MVT, there is thus a number c in $(3,5)$ where

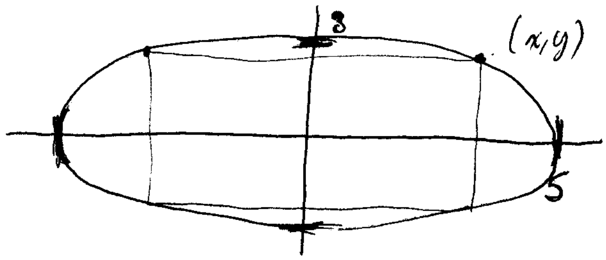
(+2) $$f'(c) = \frac{f(5)-f(3)}{5-3} = \frac{(5-4)^2-(3-4)^2}{2} = \frac{1-1}{2} = 0,$$

(c) Let $g(x) = (x-4)^{-2}$. Show that there is no value of c in the interval $(3,5)$ such that $g(5) - g(3) = g'(c)(5-3)$, and explain why this does not contradict the Mean Value Theorem.

(+2) $g'(x) = -2(x-4)^{-3}$, and $g(5)-g(3) = (5-4)^{-2}-(3-4)^{-2} = 0$,
Thus we seek c where $g'(c)=0$, But $g'(c) = \frac{-2}{(x-4)^{-3}}$ is never 0, so there is no c in $(3,5)$ where
$g(5)-g(3) = g'(c)(5-3)$.

(+3) This does not contradict the Mean Value Theorem because $g(x)$ is not continuous at $x=4$ and thus not on the interval $[3,5]$. The Mean Value theorem therefore does not apply.

(10) Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$, where a and b are positive constants. (You may assume that the rectangle with largest area has sides which are parallel to the axes.)



(+2) The area of the inscribed rectangle is given by $A = (2x)(2y) = 4xy$, where $x \geq 0$, $y \geq 0$ satisfy $\frac{x^2}{25} + \frac{y^2}{9} = 1$ as in the picture. Also,

(+2) $$\frac{x^2}{25} + \frac{y^2}{9} = 1 \Rightarrow y = \frac{\sqrt{225 - 9x^2}}{5}.$$

(+2) Thus $A(x) = 4 \cdot x \cdot \frac{\sqrt{225-9x^2}}{5} = \frac{4}{5} \cdot x \cdot \sqrt{225 - 9x^2}$

(+3) Also, $0 \leq x \leq 5$, so we use the Closed Interval Method.

Critical values:

$$A'(x) = \frac{4}{5}\sqrt{225-9x^2} + \frac{4}{5} x \cdot \frac{1}{2}(225-9x^2)^{-1/2}(-18x)$$

$$= \frac{4}{5}\sqrt{225-9x^2} - \frac{36}{5} x^2 \cdot \frac{1}{\sqrt{225-9x^2}}.$$

(+2) $\sqrt{225-9x^2} = 0$ only at $x = \pm 5$ so f is differentiable in the interval $(0,5)$. Also, $A'(x) = 0$ when

$$0 = \frac{5}{4}\left(\sqrt{225-9x^2}\right) A'(x) = (225 - 9x^2) - 9x^2 = 225 - 18x^2$$

$$x^2 = \frac{225}{18}, \text{ or } x = \sqrt{\frac{225}{18}} = \frac{15}{3\sqrt{2}} = \frac{5}{\sqrt{2}}.$$

Check endpoints and critical values: $A(0) = A(5) = 0$, (+1)

$$A\left(\frac{5}{\sqrt{2}}\right) = \frac{4}{\cancel{5}} \frac{\cancel{5}}{\sqrt{2}} \cdot \sqrt{225 - 9 \cdot \frac{25}{2}} = \frac{4}{\sqrt{2}} \cdot \sqrt{225 - \frac{225}{2}} = \frac{4}{\sqrt{2}} \cdot \sqrt{\frac{225}{2}} = \frac{4}{2} \cdot 15 = 30$$ (+2)

Area = 30