MA 113 — Calculus I
Exam 4

Fall 2007
December 11, 2007

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please:

1. check answers when possible,
2. clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
3. give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each question is followed by space to write your answer. Please write your solutions neatly in the space below the question. You are not expected to write your solution next to the statement of the question.

You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Notice that there is an additional bonus problem. For its solution you can earn up to 10 points for extra credit.

Name: ______________________________

Section: __________

Last four digits of student identification number: __________

Secs 1-11	Secs 12-23	Question	Score	Total
Jiang		1		6
Miller		2		6
Wen	Huber	3		7
Anewall	Mahrey	4		9
Makarjan	Hineman	5		9
Croft	Hardy	6		8
Hewitt		7		7
Barra	Walker	8		9
Benando	Shroodtbeck	9		8
each TA his section(s)		Extra Credit		10
Nagel	Fleming-Larsson	10		14
Anton		11		14
Sallaye	Harris	12		14
		Free	3	3
		Total		

(1) Determine the following limits.

(a)

$$\lim_{x\to\infty} \frac{4x^3 + x^2 + \sqrt{x}}{1 - x^3}.$$

$$= \lim_{x\to\infty} \frac{4 + x^{-1} + x^{-\frac{5}{2}}}{x^{-3} - 1}$$ (2) pt

$$= \frac{4}{-1} = -4$$ (1) pt

(b)

$$\lim_{x\to 3} \frac{x^2 - 8x + 15}{x^2 - 9}.$$

$$= \lim_{x\to 3} \frac{(x-3)(x-5)}{(x-3)(x+3)}$$ (2 pt)

$$= \frac{-2}{6} = -\frac{1}{3}$$ (1pt)

(a) Answer to Q.1(a) -4

(b) Answer to Q.1(b) $-\frac{1}{3}$

(2) Find the equation of the tangent line to the curve:

$$x^2 + xy + y^2 = 7 \text{ at the point } P(1, 2).$$

Be sure to simplify your final answer in the form $y = mx + b$.

$$2x + y + xy' + 2yy' = 0$$

(1) pt $(2x + y) + y'(x + 2y) = 0$

(1) pt $y' = -\dfrac{2x + y}{x + 2y}$

(2) pt At $(1,2)$ $y' = -\dfrac{2(1) + 2}{1 + 2(2)} = -\dfrac{4}{5}$

(1) pt $y - 2 = -\dfrac{4}{5}(x - 1)$

(1) pt $y = -\dfrac{4}{5}x + \dfrac{4}{5} + 2 = -\dfrac{4}{5}x + \dfrac{14}{5}$.

The last point may be awarded if there is a minor mistake in simplication.

The equation of the tangent line is

$y =$ $-\dfrac{4}{5}x + \dfrac{14}{5}$

(3) Let f be a function defined on the interval $(0, \infty)$ by

$$f(x) = \begin{cases} \pi \sin(x) & \text{if } 0 \le x < \frac{\pi}{6} \\ 2x + c & \text{if } \frac{\pi}{6} \le x \end{cases}$$

where c is a real number.

(a) Determine a value of c such that f is continuous on [0, ∞), if such c exists.

(1) pt The function is continuous on $(0, \frac{\pi}{6})$ and $(\frac{\pi}{6}, \infty)$ by known properties of sin and polynomials.

(1) pt At $\frac{\pi}{6}$ $\lim_{x \to \frac{\pi}{6}^-} f(x) = \pi \sin(\frac{\pi}{6}) = \frac{\pi}{2}$

(1) pt $\lim_{x \to \frac{\pi}{6}^+} f(x) = 2\frac{\pi}{6} + c = \frac{\pi}{3} + c$

(2) pt $f(\frac{\pi}{6}) = \frac{\pi}{3} + c$. For continuity at $\frac{\pi}{6}$ we need $\frac{\pi}{2} = \frac{\pi}{3} + c$

$c = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

(b) Determine where the resulting function is differentiable.

At $x = \frac{\pi}{6}$ $f(x)$ is not differentiable since the derivative from left $\left(\lim_{h \to 0^-} \frac{f(\pi/6 + h) - f(\pi/6)}{h} = \pi \cos(\frac{\pi}{6}) \right)$ while derivative from right $\left(\lim_{h \to 0^+} \frac{f(\frac{\pi}{6} + h) - f(\frac{\pi}{6})}{h} = 2 \right)$ and $\pi \cos(\frac{\pi}{6}) \ne 2$.

(2) pt They may not write all this. Take one point off for no explanation or wrong explanation.

(a) **Answer Q.3(a):** $c =$ $\frac{\pi}{6}$

(b) **Answer Q.3(b):** Intervals where f is differentiable are: $(0, \frac{\pi}{6})$ and $(\frac{\pi}{6}, \infty)$

It is ok to describe the set $(0, \infty) \setminus \{\frac{\pi}{6}\}$ in other fashion!

(4) Find the derivatives of the following functions:

(a)

$$f(x) = \sqrt{x^3 + 4x^2 + 4}.$$

(3) pt

$$f'(x) = \frac{3x^2 + 8x}{2\sqrt{x^3 + 4x^2 + 4}}$$

Reduce one pt. for each mistake - until three points are gone!

(b)

$$g(x) = \frac{\sin(x^2)}{x^2}.$$

(3) pt

$$g'(x) = \frac{2x\cos(x^2)x^2 - \sin(x^2)\,2x}{x^4}$$

$$= 2\left(\frac{x^2\cos(x^2) - \sin(x^2)}{x^3}\right)$$

This is enough. Reduce one point for each mistake as above.

(c)

$$h(x) = \int_0^x \left(\frac{(1-t^2)^3}{\sqrt{t^4+t}}\right) dt.$$

(3) pt

$$\frac{(1-x^2)^3}{\sqrt{x^4+x}}$$

Zero points for any attempts at derivative. One point off for each substitution mistake.

(a) Answer to Q.4(a) $f'(x) =$__________

(b) Answer to Q.4(b) $g'(x) =$__________

(c) Answer to Q.4(c) $h'(x) =$__________

Do not take points off for copying mistakes

(5) Consider the function f defined on the interval $[-3, 3]$ by the formula

$$f(x) = x^3 - 3x^2 + 14.$$

(a) Determine all the critical numbers for f in the open interval $(-3, 3)$.

① pt $f'(x) = 3x^2 - 6x = 3x(x-2)$

$f'(x)$ is always defined.

② pt $f'(x) = 0 \Longrightarrow$ $x = 0, \quad x = 2$ c.n.

(b) Determine the locations of all the local extrema of f on the closed interval $[-3, 3]$.

② pt At $x = 0$ $f''(x) = 6x - 6 = -6 < 0$.

So $x = 0$ is a local max

At $x = 2$ $f''(x) = 6x - 6 = 6 > 0$

So $x = 2$ is

(c) What are the absolute extremum values of f on $[-3, 3]$. Determine all the corresponding values of x where these are attained.

④ pt

	x	$f(x)$
Abs. min →	-3	$(-3)^3 - 3((-3)^2) + 14 = -54 + 14 = -40$
	0	14
	2	$2^3 - 3(2^2) + 14 = 22 - 12 = 10$
Abs max	3	$(3)^3 - 3((3)^2) + 14 = 14$

(a) **Critical numbers for f in $(-3, 3)$:** 0, 2

(b) **Local maxima of f on $[-3, 3]$ are at:** 0 **Local minima are at:** 2

(c) **Absolute maximum value of f is** ~~0~~ 14 **It is attained at** 0, 3

Absolute minimum value of f is −40 **It is attained at** −3

If they have correct answers above, do not be harsh about copying mistakes.

(6) Consider the function f defined by

$$f(x) = \int_0^x \frac{dt}{t^2+t+1}.$$

(a) Determine the first two derivatives $f'(x)$ and $f''(x)$.

$f'(x) = \frac{1}{x^2+x+1}$ $f''(x) = -\frac{(2x+1)}{(x^2+x+1)^2}$

(1 pt.) (2 pt.)

(b) Using your results in (a) or other arguments, determine the interval(s) on which f is an increasing function.

f is increasing if $f'(x) > 0$

(2 pt.) x^2+x+1 is never zero and always positive (at $x=0$ it is!)

So always increasing.

Any reasonable explanation is OK. 1 pt. for just the answer.

(c) Using your results in (a) or other arguments, determine the interval(s) on which the graph of $f(x)$ is concave upward.

$f''(x)$ is positive exactly when

(1 pt for some explanation.) $-(2x+1) > 0$

since the denominator is always positive.

(2 pt) $-(2x+1) > 0 \iff 2x+1 < 0 \iff x < -\frac{1}{2}$

(a) f'(x) = ______, f''(x) = ______

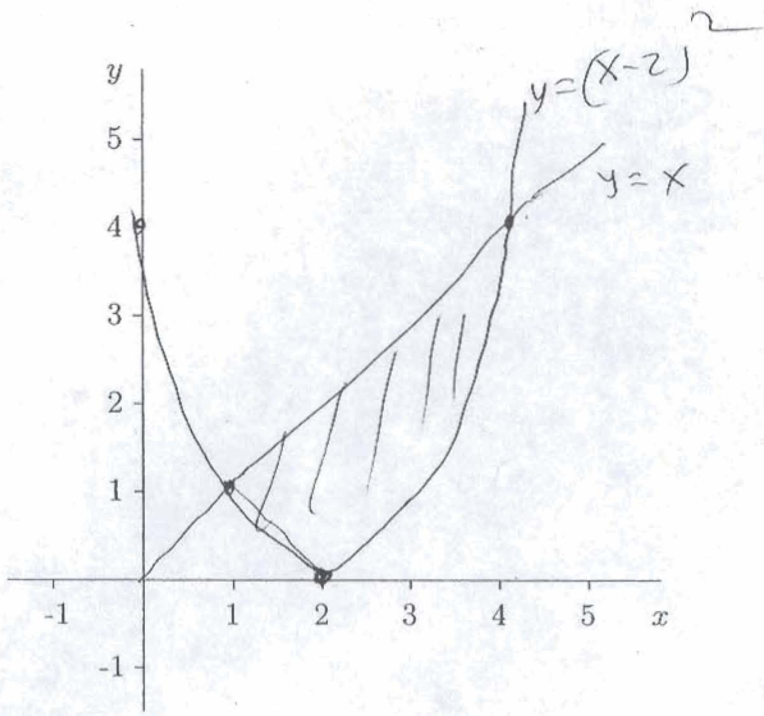
(b) Interval(s) where f is increasing: $(-\infty, \infty)$ Or always.

(c) Interval(s) where the graph of f is concave upward: $(-\infty, -\frac{1}{2})$

(7) Sketch the region enclosed between the graphs of the two curves:

$$y = (x-2)^2 \text{ and } y = x$$

and compute its area.



(2) pt

Common points: $(x-2)^2 = x$

$x^2 - 4x + 4 = x$

(3) pt

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4 \quad y = 4$

$x = 1 \quad y = 1$

(2) pt

$$\text{Area} = \int_1^4 \left(x - (x-2)^2\right) dx$$

$$= \int_1^4 (-x^2 + 5x - 4)\, dx$$

$$= -\frac{x^3}{3} + 5\frac{x^2}{2} - 4x \Big|_1^4 = \frac{9}{2}$$

The area of the region is: ______________

(8) Calculate the following integrals. For indefinite integrals, be sure to add the arbitrary constant c.

(a)

$$I = \int 6x^2(1+2x^3)^4\,dx.$$

③ pt

$u = 1 + 2x^3$

$du = 6x^2\,dx$

(2pt)

$I = \int u^4\,du$

$= \frac{u^5}{5} + c$

$= \frac{(1+2x^3)^5}{5} + c$

(1 pt)

(b)

$$I = \int \frac{t}{(4t^2+5)^2}\,dt.$$

③ pt

$u = 4t^2 + 5$

$du = 8t\,dt$

$\frac{du}{8} = t\,dt$

(2pt)

$I = \int \frac{1}{8}\frac{du}{u^2}$

$= \frac{1}{8}\frac{u^{-1}}{-1} + c$

$= -\frac{1}{8}\frac{1}{(4t^2+5)} + c$

(1 pt)

(c)

$$I = \int_0^{\pi/6} \sin(3x)\cos^2(3x)\,dx.$$

③ pt

(1pt)

$\cos(3x) = u$

$-3\sin(3x)\,dx = du$

$\sin(3x)\,dx = -\frac{1}{3}du$

(1pt)

$x = 0 \quad u = 1$

$x = \frac{\pi}{6} \quad u = \cos\left(\frac{3\pi}{6}\right) = 0$

$I = \int_1^0 \left(-\frac{1}{3}\right) u^2\,du$

$= -\frac{1}{3}\frac{u^3}{3}\Big|_1^0 = 0 - \left(-\frac{1}{9}\right)$

$= \frac{1}{9}$

(1 pt)

(a) Answer to Q.8(a) ________________

(b) Answer to Q.8(b) ________________

(c) Answer to Q.8(c) ________________

(9) Consider the integral

$$\int_1^2 \frac{1}{x}\, dx.$$

Answer the following questions.

(a) Estimate the integral by the Riemann sum obtained by using a subdivision into three equal parts and using the right endpoints as sample points.
Be sure to write all the details and to write the final answer as a single fraction.

(4 pt)

$a = 1$

$b = 2$

$\Delta = \frac{2-1}{3} = \frac{1}{3}$

$a = x_0 = 1$

$x_1 = 1 + \frac{1}{3} = \frac{4}{3}$

$x_2 = 1 + \frac{2}{3} = \frac{5}{3}$

$b = x_3 = 1 + \frac{3}{3} = 2$

$f(x_1) = \frac{3}{4}$

$f(x_2) = \frac{3}{5}$

$f(x_3) = \frac{1}{2}$

(2 pt)

~~1 pt~~

$R.S. = \frac{1}{3}\left(\frac{3}{4} + \frac{3}{5} + \frac{1}{2}\right)$

(1 pt)

$= \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{30 + 24 + 20}{120} = \frac{74}{120}$

$= \frac{37}{60}$

~~1 pt~~

(b) Determine if your estimate is less than or bigger than the actual value of the integral. Explain.

Lower since the function is decreasing.

~~1 pt~~

(1 pt)

(a) Answer to Q.9(a) $74/120 = \frac{37}{60} \doteq 0.617$

(b) Answer to Q.9(b) ________________

Extra Credit Problem.

Determine if the following statements are true or false. No justification is required.

(a) Suppose the function f is continuous on $[1,5]$ and differentiable on $(1,5)$.
If $f(1) = -5$ and $f(5) = 20$, then $f'(t) \geq 5$ for at least one t in $(1,5)$.

2pts each

Answer: True [✓] **False** [ ]

(b) Suppose the function f is continuous on the set of real numbers.
Then f is differentiable.

Answer: True [ ] **False** [✓]

(c) Suppose that f is a continuous function on $[1,5]$ and $f'(2) = 0$. Then f has a local extremum at 2.

Answer: True [ ] **False** [✓]

(d) Let f be a function that is twice differentiable on the set of real numbers.
If $f''(2) = 0$ then f has an inflection point at 2.

Answer: True [ ] **False** [✓]

(e) If f is a continuous function on the closed interval $[1,10]$, then f has an absolute maximum at some point c in $[1,10]$.

Answer: True [✓] **False** [ ]

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(10) (a) Carefully state the Fundamental Theorem of Calculus (both parts).

(FTC) Suppose that f is continuous on [a, b]. Then the function $g(x)=\int_a^x f(t)\,dt$ is differentiable on (a, b) with $g'(x) = f(x)$.

Moreover, if F is any antiderivation of f, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

3pt + 2pt

(b) Consider the function f defined by $f(x) = \int_1^x \frac{dt}{1+t^2}$. Determine the derivatives of $f(x)$ and $h(x) = f(\frac{1}{x})$.

$$f'(x) = \frac{1}{1+x^2}$$

2pt

$$h'(x) = \frac{d}{dx}\int_1^{\frac{1}{x}} \frac{dt}{1+t^2} = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = -\frac{1}{1+x^2}$$

3pt

(c) Consider the function g on $(0, \infty)$ defined by $g(x) = f(x) + f(\frac{1}{x})$. Using your results in (b) or other arguments, argue that g is a constant function.

$$g'(x) = \frac{1}{1+x^2} + \left(-\frac{1}{1+x^2}\right) = 0$$

so g is a constant function.

2pt

(d) Determine $g(1)$.

$$g(1) = f(1) + f(1) = 0 + 0 = 0$$

2pt

Derivative of $f(x)$: __________

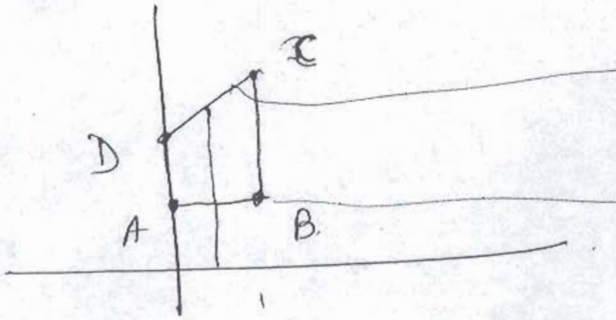
Derivative of $f(\frac{1}{x})$: __________

$g(1) =$ 0

(11) Consider the quadrangle whose vertices are $A(0,1)$, $B(1,1)$, $C(1,3)$, $D(0,2)$.

(a) Make a rough sketch of the quadrangle.

5 pt



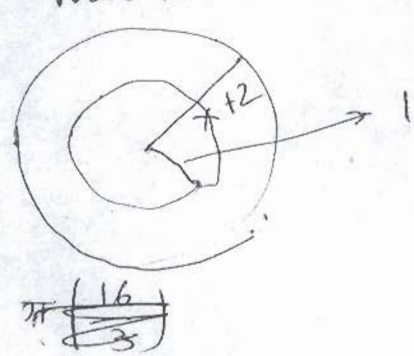
(b) Set up the integral to determine the volume obtained by rotating the quadrangle about the x-axis.

4 pt

Washer.

$$\int_0^1 \pi\left((x+2)^2 - 1^2\right)dx$$

$$= \int_0^1 \pi\left(x^2+4x+3\right)dx$$



(c) Evaluate the integral and determine the volume.

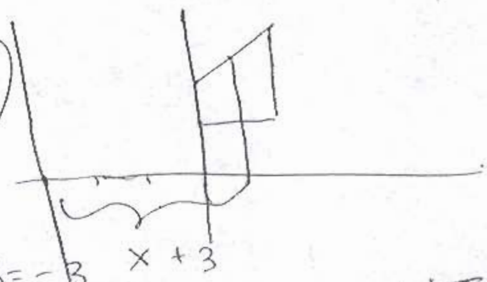
2 pt

$$\left(\pi \; \frac{x^3}{3} + 2x^2 + 3x \Big|_0^1\right) = \pi\left(\frac{1}{3} + 2 + 3\right) = 16\pi/3$$

(d) If the same quadrangle is rotated about the line $x = -3$, write down the integral that computes the volume of the resulting solid of revolution. You are not asked to evaluate this new integral.

3 pt

Cylindrical shells



$$\int_0^1 2\pi(x+3)\left((x+2)-1\right)dx$$

$$= \int_0^1 2\pi(x+3)(x+1)\,dx$$

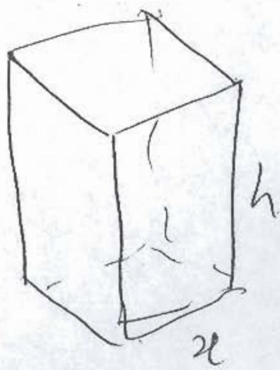
(c) **Volume:** $16\pi/3$

(d) **Integral:** ______________________

(12) A company wishes to manufacture a strong box with a square base and a volume of 625 cubic inches.

It is given that the sides of the box cost \$1 per square inch to manufacture and the top and the bottom cost \$5 per square inch to manufacture.

Find the dimensions and the cost of the cheapest such box.



Let x be the length of a side of the base and h the height. $0 \le x$

The volume

$V = hx^2 = 625$ Given

(7 pt correct setup)

$h = \frac{625}{x^2}$.

$\text{Cost} = (4hx)1 + (2x^2)5$

$C(x) = \frac{(4)(625)}{x^2}x + 10x^2$ — Cost function

$= \frac{2500}{x} + 10x^2$

(2pt) → Minimize $C(x)$ on $(0, \infty)$

$C'(x) = 0 \Rightarrow -\frac{2500}{x^2} + 20x = 0$ $\quad 20x^3 = 2500$ $\quad x^3 = 125$ so $x = 5$

(2pt) → This is the only c.n. on $(0,\infty)$. $C(x) \to \infty$ at both end points $(0, \infty)$. So $x = 5$ is abs. min

(3pt) Height of the cheapest box: 25 in

Base side length of the cheapest box: 5 in

Cost of the cheapest box: 750 \$.

$C(5) = \frac{2500}{5} + 10(25)$ $\qquad h = \frac{625}{5^2} = 25$

→ Evaluations and final ans