Calculus I
Final

Common exam
1 May 2006

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer **(unsupported answers may not receive credit).** 3) give exact answers, rather than decimal approximations to the answer.
Each question is followed by space to write your answer. Please lay out your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name ______________________

Section ___________

Last four digits of student identification number ___________

Question	Score	Total
p. 1		14
p. 2		14
p. 3		14
p. 4		14
p. 5		14
Q 10		14
Q 11		14
Q 12		14
Free	2	2
		100

Burns (1-8) Zhang (9
Nie (1-8) Zhang (9
Namwani (1-8) Erickson
Guo (1-8) Erickson
Cox
Hislop
Brown
Nagel

1. Find the equation of the tangent line to the graph of $f(x) = x^3 + 2x^2 + 2x$ at $x = -1$. Put your answer in the form $y = mx + b$.

2 2

The derivation of $f(x)$ is $f'(x) = 3x^2 + 4x + 2$, thus $f'(-1) = 1$ and $f(-1) = -1$. Hence the equation of the tangent line is:

1

$$y - f(-1) = f'(-1)\,(x - (-1)), \quad 2$$

so $\quad y + 1 = x + 1,$

i.e. $\quad y = x.$ $\quad$ 1

$y =$ x

2. Find the limits.

(a) $\lim_{x\to 0} \frac{\sin^2 x}{5x^2}$. $\quad = \frac{1}{5}\left[\lim_{x\to 0} \frac{\sin x}{x}\right]^2 = \frac{1}{5}$ $\quad$ 3

(b) $\lim_{x\to\infty} \sin\left(\frac{\pi}{2} + \frac{4}{x}\right)$

1

Since sin is continuous, we get

$$\lim_{x\to\infty} \sin\left(\frac{\pi}{2} + \frac{4}{x}\right) = \sin\left(\lim_{x\to\infty}\left(\frac{\pi}{2} + \frac{4}{x}\right)\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$ 3

(a) $\frac{1}{5}$, (b) 1

3. Suppose that f is defined by

$$f(x) = \begin{cases} \dfrac{x^2+3x+2}{x+2}, & \text{for } x \neq -2 \\ A, & x = -2 \end{cases}$$

Find A so that f is continuous at $x = -2$.

If $x \neq -2$, then $\dfrac{x^2+3x+2}{x+2} = \dfrac{(x+2)(x+1)}{x+2} = x+1$

whose limit is $-2+1=-1$ as x approaches -2, i.e.

$\lim\limits_{x \to -2} f(x) = -1$. Hence A must be -1.

$A =$ -1

4. Find the linear approximation $L(x)$ to $f(x) = \sqrt{1+2x}$ at $x = 4$. Use this linear approximation to compute an approximate value for $f(3.7)$.

2

We compute $f(4) = 3$, $f'(x) = \frac{1}{2}(1+2x)^{-\frac{1}{2}} \cdot 2 = \dfrac{1}{\sqrt{1+2x}}$, so

$f'(4) = \frac{1}{3}$. Hence $L(x) = f(4) + f'(4)(x-4) = 3 + \frac{1}{3}(x-4)$

and $f(3.7) \approx L(3.7) = 3 + \frac{1}{3}(-0.3) = 2.9$ 2

$L(x) =$ $3 + \frac{1}{3}(x-4)$, $f(3.7) \approx$ 2.9

5. Let $f(x) = \dfrac{1-x}{1+x}$. Compute the derivative, $f'(x)$, and use the derivative to find the intervals where f is increasing or decreasing.

3 1

The quotient rule gives $f'(x) = \dfrac{-(1+x)-(1-x)}{(1+x)^2} = -\dfrac{2}{(1+x)^2} < 0.$

1

Since f (and f') are not defined at $x=-1$, f is decreasing on $(-\infty,-1)$ and $(-1,\infty)$

$f'(x) =$ $-\dfrac{2}{(1+x)^2}$

Intervals of decrease $(-\infty,-1)$, $(-1,\infty)$ 2

Intervals of increase ____________

6. Find the absolute maximum value and absolute minimum value of the function $f(x) = \dfrac{1}{1+x^2}$ on the interval $[2, 5]$.

3 1 1

$f'(x) = -\dfrac{2x}{(1+x^2)^2} = 0$ iff $x=0$ which is not in $[2,5]$.

Since f is continuous on $[2,5]$, it suffices to compute

x	2	5
$f(x)$	$\frac{1}{5}$	$\frac{1}{26}$
	max	min

2

Absolute maximum value $\frac{1}{5}$, absolute minimum value $\frac{1}{26}$

7. Find the derivatives. (a) $\frac{d}{dx}\int_2^x \sin(t^2)\,dt$, (b) $\frac{d}{dx}\int_x^4 \cos(t^2)\,dt.$ $= \frac{d}{dx}\left[-\int_4^x \cos(t^2)dt\right].$

Hence the Fundamental Theorem of Calculus immediately gives

3

4

(a) $\sin(x^2)$, (b) $-\cos(x^2)$

8. Suppose that a particle is moving along the x-axis so that after t seconds, the particle is $x(t)$ meters to the right of the origin. The acceleration of the particle is 12 meters/second^2 for all t. At time $t = 0$, the particle is located at the origin and at time $t = 2$ seconds, the velocity of the particle is -40 meters/second. Find the position function $x(t)$.

We are given that $x(0) = 0$, $x'(2) = -40$, and $\frac{d^2x}{dt^2} = x''(t) = 12$. 1 1 1

Hence we get $x'(t) = 12t + C$, 1 thus $-40 = x'(2) = 12 \cdot 2 + C$,

therefore $C = -64$ and $x'(t) = 12t - 64$. 1

Now it follows that $x(t) = 6t^2 - 64t + D$. 1

Using $0 = x(0) = D$ we obtain: 1

$x(t) =$ $6t^2 - 64t$

9. Evaluate the following integrals.

5 (a) $\displaystyle\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx$ $= \sqrt{x^2+9}\,\Big|_0^4$

4 (b) $\displaystyle\int_{-2}^2 x\sin(x^2)\,dx$

5 (c) $\displaystyle\int_1^4 \frac{x^2+2}{x^2}\,dx$ $= \displaystyle\int_1^4 1 + \frac{2}{x^2} = \left(x - \frac{2}{x}\right)\Big|_1^4$

5 (a) Using $u = x^2+9$, thus $\frac{du}{dx} = 2x$, we get

$$\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \int_9^{25} \frac{1}{2}\frac{1}{\sqrt{u}}\,du = \sqrt{u}\,\Big]_9^{25} = 5-3 = 2.$$

4 (b) Using $u = x^2$, thus $\frac{du}{dx} = 2x$, we obtain

$$\int_{-2}^{2} x\cdot\sin(x^2)\,dx = \int_4^4 \frac{1}{2}\sin(u)\,du = 0.$$

(Alternatively, this follows since $f(x) = x\cdot\sin(x^2)$ is an odd function.)

5 (c) $$\int_1^4 \frac{x^2+2}{x^2}\,dx = \int_1^4 (1+2x^{-2})\,dx = \left[x - \frac{2}{x}\right]_1^4 = 4 - \frac{1}{2} - 1 + 2 = \frac{9}{2}.$$

(a) $\displaystyle\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx =$ 2

(b) $\displaystyle\int_{-2}^2 x\sin(x^2)\,dx =$ 0

(c) $\displaystyle\int \frac{x^2+2}{x^2}\,dx =$ $\frac{9}{2}$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

10. (a) Give the definition of the derivative of a function f at a point x. 5

(b) Let $g(x) = \dfrac{1}{1+2x}$. Use the definition to compute g', the derivative of g. 5

(c) Give the domain of $g'(x)$. 4

(a) The derivative of f at x is $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided this limit exists.

(b) $$g'(x) = \lim_{h \to 0} \frac{\frac{1}{1+2(x+h)} - \frac{1}{1+2x}}{h} = \lim_{h \to 0} \frac{1+2x - (1+2x+2h)}{(1+2x+2h)(1+2x)\cdot h}$$

$$= \lim_{h \to 0} \frac{-2h}{(1+2x+2h)(1+2x)\cdot h} = \lim_{h \to 0} \frac{-2}{(1+2x+2h)(1+2x)}$$

$$= -\frac{2}{(1+2x)^2}.$$

(c) g (and g') are not defined if $1+2x = 0$, i.e. $x = -\frac{1}{2}$.

Hence we get from part (b) that the domain of g' is

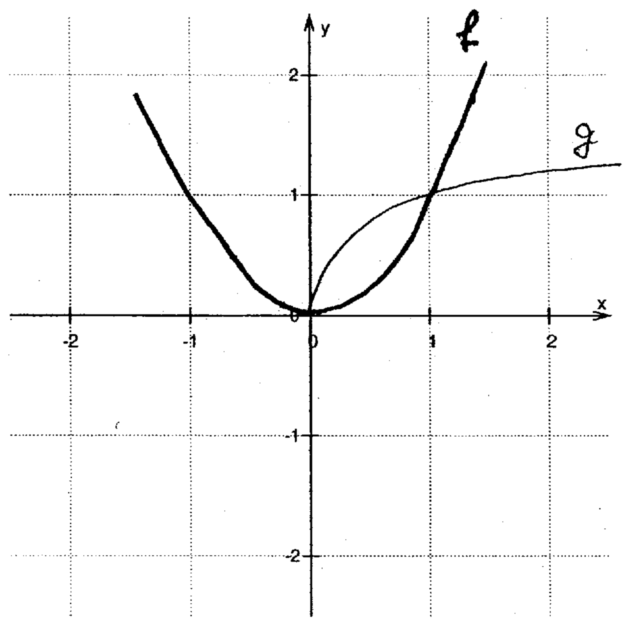
$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty) = \{x \in \mathbb{R} \mid x \neq -\frac{1}{2}\}$.

11. Let $f(x) = x^2$ and $g(x) = \sqrt{x}$.

(a) Find the points where the graphs of f and g intersect. (3)

(b) Make a rough sketch of the graphs of f and g. (4)

(c) Find the area enclosed by the graphs of f and g. (5)



(a) f and g intersect iff $f(x) = g(x)$, i.e. $x^2 = \sqrt{x}$, so

$x^4 = x$ or $0 = x^4 - x = x(x^3-1) = x(x-1)\underbrace{(x^2+x+1)}_{>0}$

which implies $x = 0$ or $x = 1$.

Indeed, f and g intersect at $(0,0)$ and at $(1,1)$.

(c) The area A is:

$$A = \int_0^1 |f(x) - g(x)|\,dx = \int_0^1 [\sqrt{x} - x^2]\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \Big]_0^1$$

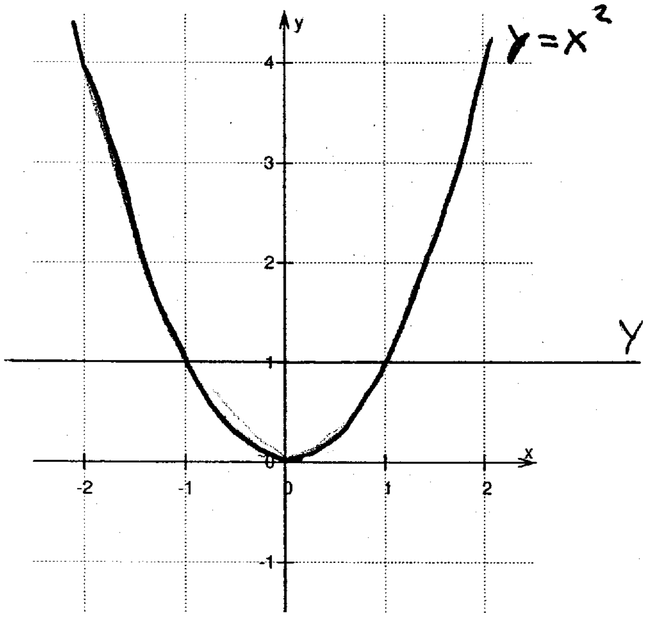
$$= \frac{2}{3} - \frac{1}{3} = \underline{\underline{\frac{1}{3}}}.$$

12. Let R be the region enclosed by the parabola $y = x^2$ and the line $y = 1$. Let S be the solid of revolution obtain by rotating the region R about the line $y = 3$.

(a) Sketch the region, R.

(b) Write an integral which gives the volume of S.

(c) Evaluate the integral from part (c).



(b) The two curves intersect when $x^2=1$, i.e. $x=1$ or $x=-1$.

The inner radius of S is $3-1=2$ and the outer radius is $3-x^2$. Hence the volume V of S is

$$V = \pi\int_{-1}^{1} \left[(3-x^2)^2 - 2^2\right] dx$$

(c)

$$= \pi \int_{-1}^{1} (5-6x^2+x^4)\,dx$$

$$= \pi \left[5x - 2x^3 + \tfrac{1}{5}x^5\right]_{-1}^{1}$$

$$= \pi\left[10 - 4 + \tfrac{2}{5}\right]$$

$$= \underline{\underline{\frac{32}{5}\pi}}.$$