Calculus I
Examination 4

3 May 2007

Answer all of the following questions. Additional sheets are available if necessary. No books or notes may be used. You may use a calculator. You may not use a calculator which has symbolic manipulation capabilities. Please: 1) check answers when possible, 2) clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*, 3) give exact answers, rather than decimal approximations to the answer.
Please write your solutions neatly in the space below the question. You are not expected to write each solution next to the statement of the question. If you use the back of a sheet, indicate this by the question.
You are to answer two of the last three questions. Please indicate which problem is not to be graded by crossing through its number on the table below.

Name Solutions

Section __________

Last four digits of student identification number __________

Question	Score	Total
p. 2/Q1		14
p. 3/Q2–3		14
p. 4/Q4–5		14
p. 5/Q6		14
p. 6/Q7–8		14
p. 7/Q9		14
p. 8/Q10		14
p. 9/Q11		14
Free	2	2
		100

1. Find the derivatives. In part (b), you must simplify your answer.

(a) $f(x) = \sqrt{x^2 + 3x}$ (b) $g(x) = \dfrac{x^2 - 1}{x^2 + 1}$ (c) $h(x) = x \sin(x^2)$.

2 pts $$f'(x) = \frac{d}{dx}(x^2+3x)^{1/2} = \frac{1}{2}(x^2+3x)^{\frac{1}{2}-1}\frac{d}{dx}(x^2+3x)$$ by generalized power rule

2 pts $$= \frac{1}{2}(x^2+3x)^{-1/2}(2x+3)$$

2 pts $$g'(x) = \frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$ by quotient rule

2 pts $$= \frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

2 pts $$h'(x) = \sin(x^2)\frac{d}{dx}(x) + x\frac{d}{dx}\sin(x^2)$$ by product rule

2 pts $$= \sin(x^2) + x\cos(x^2)\frac{d}{dx}(x^2)$$ by chain rule

2 pts $$= \sin(x^2) + 2x^2\cos(x^2)$$

(a) $f'(x) =$ $\frac{1}{2}(x^2+3x)^{-\frac{1}{2}}(2x+3)$, (b) $g'(x) =$ $\dfrac{4x}{(x^2+1)^2}$

(c) $h'(x) =$ $\sin(x^2) + 2x^2\cos(x^2)$

2. An object is dropped from a height of 180 meters. Find the speed when it hits the ground. Use that the acceleration of gravity is 10 meters/second^2 in the downward direction.

1pt Initial height $h(0) = 180$

1pt Initial velocity $h'(0) = 0$

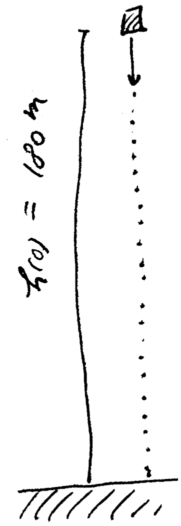
1pt Acceleration $h''(t) = -10$

1pt Velocity $h'(t) = -10t + h'(0) = -10t$

1pt Height $h(t) = -10\frac{t^2}{2} + h(0) = -5t^2 + 180$

1pt Time elapsed $-5t^2 + 180 = 0 \quad t^2 = \frac{180}{5} = 36 \quad t = 6$

1pt Ground velocity $h'(6) = -10 \cdot 6 = -60 m/s$



60 m/s

3. Find the equation of the line which is perpendicular to the graph of $f(x) = \frac{1}{x}$ at $x = 2$.

2 pts Graph slope $f'(2) = \frac{d}{dx}\left(\frac{1}{x}\right)\Big|_{x=2} = -\frac{1}{x^2}\Big|_{x=2} = -\frac{1}{4}$

2 pts Negative reciprocal $-\frac{1}{-\frac{1}{4}} = 4$

1 pt Point of contact $x = 2$

$y = \frac{1}{x}\Big|_{x=2} = \frac{1}{2}$

2 pts Perpendicular line $y - \frac{1}{2} = 4(x-2)$

$y - \frac{1}{2} = 4(x-2)$

4. Use calculus to find all values of x so that $f(x) = x^5 - 5x$ has a local maximum or minimum at x.

1 pts — 1^{st} derivative — $f'(x) = 5x^4 - 5$

2 pts — Critical points — $5x^4 - 5 = 0 \quad x^4 = \frac{5}{5} = 1 \quad x = \pm 1$

2 pts — 2^{nd} derivative test — $f''(x) = 20x^3 > 0 \quad @ \quad x = +1$ min

2 pts — $< 0 \quad @ \quad x = -1$ max

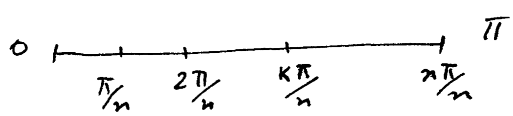
Local maximum at $x = -1$

Local minimum at $x = +1$

5. Find the limit by expressing the limit as a definite integral and evaluating the integral.

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sin(\frac{k\pi}{n})$$

2 pts — Sample points



2 pts — $\lim\limits_{n \to \infty} \frac{1}{n} \sum\limits_{k=1}^{n} \sin\left(\frac{k\pi}{n}\right) = \lim\limits_{n \to \infty} \frac{1}{\pi} \cdot \frac{\pi}{n} \sum\limits_{k=1}^{n} \sin(x) \Big|_{x = \frac{k\pi}{n}}$

2 pts — $= \frac{1}{\pi} \int_0^{\pi} \sin(x)\, dx = -\frac{1}{\pi} \cos(x) \Big|_0^{\pi}$

1 pt — $= \frac{-\cos\pi + \cos 0}{\pi} = \frac{2}{\pi}$

$\frac{2}{\pi}$

6. Evaluate. (a) $\int_0^1 (x^3 - x^{5/2})\,dx$, (b) $\int_0^{\pi/3} \cos^3(\theta)\sin(\theta)\,d\theta$, (c) $\int x\sqrt{x^2-a^2}\,dx$

$$\int_0^1 (x^3 - x^{5/2})\,dx = \frac{x^{3+1}}{3+1} - \frac{x^{5/2+1}}{5/2+1}\Big|_0^1$$

$$= \frac{1}{4} - \frac{2}{7} = -\frac{1}{28}$$

2 pts

$$\int_0^{\pi/3} \cos^3(\theta)\sin(\theta)\,d\theta = -\int_1^{1/2} u^3\,du = \int_{1/2}^1 u^3\,du$$

$$= \frac{u^4}{4}\Big|_{1/2}^1$$

$$= \frac{1}{4} - \frac{1}{4\cdot 16} = \frac{15}{64}$$

3 pts

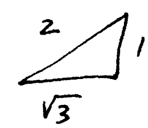
$\cos(\theta) = u$

$-\sin(\theta)\,d\theta = du$ $\qquad \sin(\theta)\,d\theta = -du$

2 pts

$\theta = \frac{\pi}{3}$ ...... $u = \cos\frac{\pi}{3} = 1/2$

$\theta = 0$ ..... $u = \cos 0 = 1$



2 pts

$$\int x\sqrt{x^2-a^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du\,\Big|_{u=x^2-a^2}$$

$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}\Big|_{u=x^2-a^2} + C$$

$$= \frac{1}{3}(x^2-a^2)^{3/2} + C$$

3 pts

$u = x^2 - a^2$

$du = 2x\,dx$ $\qquad x\,dx = \frac{1}{2}du$

2 pts

(a) $-\frac{1}{28}$, (b) $\frac{15}{64}$

(c) $\frac{1}{3}(x^2-a^2)^{3/2} + C$

7. Let R be the region which lies between the lines $x = 2$, $x = 5$, $y = 3$ and $y = 4 + x$. Let S the solid of revolution created by rotating R about the line $y = 1$. Write an integral that represents the volume of S. Do not evaluate the integral.

1 pt Washer inner radius

1 pt ..... outer ....

2 pts Cross-section area

2 pts Volume

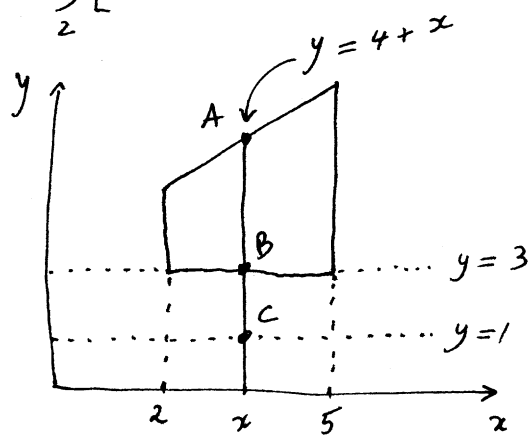
1 pt Diagram

$BC = 3 - 1 = 2$

$AC = 4 + x - 1 = 3 + x$

$A(x) = \pi(3+x)^2 - \pi(2)^2$

$\int_2^5 \left[\pi(3+x)^2 - \pi(2)^2\right] dx$



$$\int_2^5 \left[\pi(3+x)^2 - \pi(2)^2\right] dx$$

8. Find all intervals where the function

$$g(x) = \int_0^x \frac{1}{2+t^4}\, dt$$

is concave up.

2 pts $g'(x) = \dfrac{1}{2+x^4}$ by fundamental theorem

2 pts $g''(x) = \dfrac{-4x^3}{(2+x^4)^2}$ by generalized power rule

1 pt $g''(x) = 0$ for $x = 0$

2 pts

x	$(-\infty, 0)$	$(0, +\infty)$
$g''(x)$	+	−
$g(x)$	C.U.	C.D.

$(-\infty, 0)$

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

9. (a) State the definition of the derivative of a function f at a number x.

   (b) Use the definition to compute the derivative of $g(x) = \dfrac{1}{x+1}$. No credit will be given for computing the derivative with the differentiation rules. You may want to use the differentiation rules to check your answer.

2 pts The derivative of a function f at a number x is the limit $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ when this limit exists.

2 pts $$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \quad \text{by definition}$$

2 pts $$= \lim_{h \to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

2 pts $$= \lim_{h \to 0} \frac{x+1 - x - h - 1}{h(x+h+1)(x+1)} \quad \text{by algebra}$$

2 pts $$= \lim_{h \to 0} \frac{-h}{h(x+h+1)(x+1)}$$

2 pts $$= \lim_{h \to 0} \frac{-1}{(x+h+1)(x+1)}$$

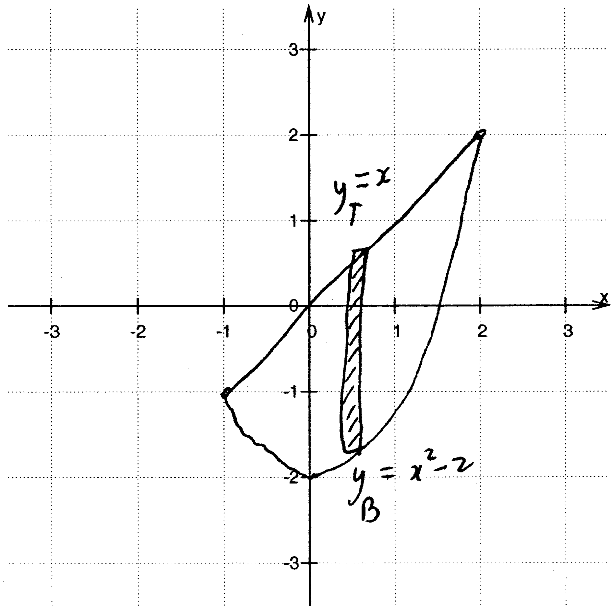
2 pts $$= -\frac{1}{(x+1)^2} \quad \text{by limit laws}$$

10. (a) Sketch the graphs of $y = x^2 - 2$ and $y = x$.

(b) Use the definite integral to write an expression which gives the area between the graph of $y = x^2 - 2$ and the graph of $y = x$.

(c) Use your answer to (b) to find the area between the graphs of $y = x^2 - 2$ and $y = x$.

4 pts



2 pts Intersection points $\begin{cases} y = x \\ y = x^2 - 2 \end{cases}$ $x^2 - 2 = x$

$x^2 - x - 2 = 0$ $(x-2)(x+1) = 0$ $x = -1, x = 2$

2 pts

2 pts Area $\int_{-1}^{2} (y_T - y_B)\, dx = \int_{-1}^{2} (x - x^2 + 2)\, dx$

2 pts $= \frac{x^2}{2} - \frac{x^3}{3} + 2x \Big|_{-1}^{2}$

1 pt $= 2 - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2$

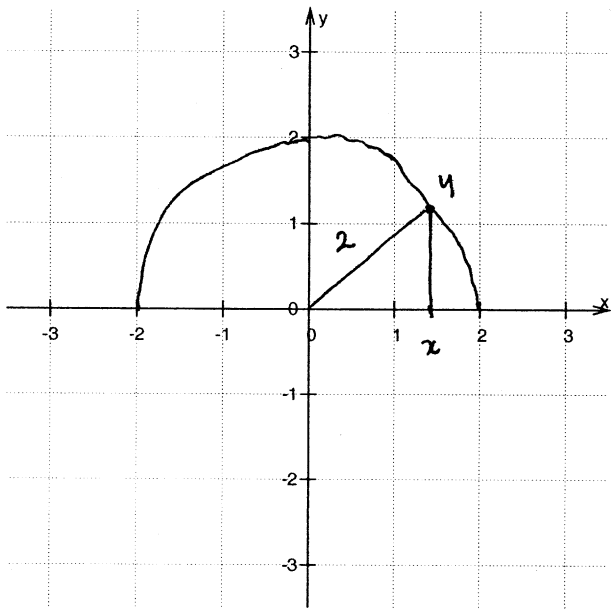
1 pt $= 4\frac{1}{2}$

11. (a) Sketch a region R so that if R is rotated about the x-axis, the resulting solid of revolution is a sphere of radius 2.

(b) Write down a definite integral which represents the volume of the sphere that we obtain when the region R in part (a) is rotated about the x-axis.

(c) Evaluate the integral in (b) to find the volume of a sphere of radius 2.

4 pts



2 pts Disk radius $\quad y = \sqrt{4-x^2}$ by Pythagorean theorem

2 pts Cross-section area $\quad A(x) = \pi\left(\sqrt{4-x^2}\right)^2 = \pi(4-x^2)$

2 pts Volume $\quad 2\int_0^2 A(x)\,dx = 2\int_0^2 \pi(4-x^2)\,dx$

by symmetry

2 pts $\quad = 2\pi\left(4x - \frac{x^3}{3}\right)\Big|_0^2$

1 pt $\quad = 2\pi\left(8 - \frac{8}{3}\right)$

1 pt $\quad = \frac{32\pi}{3}$