(1) Consider the function

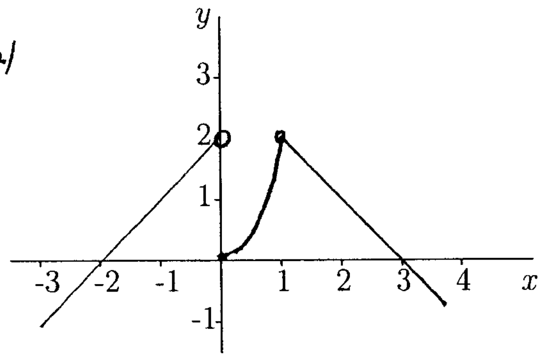
$$f(x) = \begin{cases} x+2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \le x \le 1 \\ 3-x & \text{if } 1 < x \end{cases}.$$

(a) Sketch the graph of the function f.

(b) Determine all numbers a such that f is not continuous at a.

(c) Find all numbers a such that f is not differentiable at a.

(a)



(b)+(c) Since $x+2$, $2x^2$, and $3-x$ are differentiable, it suffices to consider f at 0 and 1. At 0 we get $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} (x+2) = 2$ and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (2x^2) = 0$$

Since $0 \neq 2$, f is not continuous, thus also not differentiable at 0.
At 1 we obtain $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (2x^2) = 2$ and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x) = 3-1 = 2.$$ The one-sided limits

agree, hence f is continuous at 1. It follows that

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \frac{d}{dx}(2x^2)\Big|_{x=1} = 4x\Big|_{x=1} = 4 \text{ and}$$

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \frac{d}{dx}(3-x)\Big|_{x=1} = -1.$$ Hence f is not differentiable at 1.

(b) Discontinuities at ___0___ (c) Not differentiable at ___0 and 1___

(2) The cost in dollars of producing x units of a certain commodity is $C(x) = 5000 + 10x + \frac{1}{5}x^2$.

(a) Find the average rate of change of C with respect to x when the production level is changed from $x = 10$ to $x = 15$.

(b) Find the instantaneous rate of change of C with respect to x when $x = 10$.

(c) If the units, x, are increasing at the constant rate of 2 units per day, find the rate at which C is changing with respect to time (measured in days) when $x = 10$.

(a) $$\frac{C(15) - C(10)}{15 - 10} = \frac{1}{5}\left(10(15-10) + \frac{1}{5}(15^2 - 10^2)\right)$$
$$= 10 + 3^2 - 2^2$$
$$= \underline{\underline{15}}.$$

(b) $C'(x) = 10 + \frac{2}{5}x$, hence $C'(10) = 10 + \frac{2}{5} \cdot 10 = \underline{\underline{14}}.$

(c) Denote the time by t. It is given that $\frac{dx}{dt} = x'(t) = 2$.

Thus, it follows for $C(x(t)) = 5000 + 10x(t) + \frac{1}{5}(x(t))^2$,

$$\frac{d}{dt} C(x(t)) = 10\, x'(t) + \frac{2}{5}\, x(t) \cdot x'(t)$$
$$= 20 + \frac{4}{5}\, x(t).$$

At the time t_0 when $x(t_0) = 10$, we get

$$\left.\frac{d}{dt} C(x(t))\right|_{t=t_0} = 20 + \frac{4}{5} \cdot 10 = \underline{\underline{28}}.$$

(a) Average change of C <u>15</u>

(b) Instantaneous rate of change of C <u>14</u>

(c) Related rate of change of C <u>28</u>

(3) Determine the following limits.

(a) $\lim_{x\to 0} \frac{\cos^2(x)}{4x^3-5} = \frac{\cos^2(0)}{4\cdot 0^3 - 5} = \underline{\underline{-\frac{1}{5}}}$ Because the function is continuous at 0.

(b) $\lim_{x\to\infty} \frac{7x^4+x^2+\sqrt{x}}{1+x^4} = \lim_{x\to\infty} \frac{7+\frac{1}{x^2}+\frac{1}{x^{7/2}}}{\frac{1}{x^4}+1} = \frac{7+0+0}{0+1} = \underline{\underline{7}}.$

(c) $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x+3}{x+2} = \underline{\underline{\frac{5}{4}}}.$

(a) **Answer to Q.3(a)** $-\frac{1}{5}$

(b) **Answer to Q.3(b)** 7

(c) **Answer to Q.3(c)** $\frac{5}{4}$

(4) Determine the following limits by interpreting them as a derivative or integral.

(a) $\lim_{h\to 0} \frac{\sqrt[4]{16+h} - \sqrt[4]{16}}{h}$.

The derivative of $f(x) = \sqrt[4]{16+x}$ at 0 is $f'(0) = \lim_{h\to 0} \frac{\sqrt[4]{16+h} - \sqrt[4]{16}}{h}$.

Since $f'(x) = \frac{1}{4}(16+x)^{-\frac{3}{4}}$, we get $\lim_{h\to 0} \frac{\sqrt[4]{16+h} - \sqrt[4]{16}}{h} = f'(0) = \frac{1}{4}\cdot\left(\frac{1}{2}\right)^{-3} = \underline{\underline{\frac{1}{32}}}$.

(b) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{3 + \frac{i}{n}}$.

Using right endpoints, the Riemann sum R_n becomes

$R_n = \Delta x \sum_{i=1}^{n} f(a + i\cdot\Delta x)$, where $\Delta x = \frac{b-a}{n}$. Hence taking $\Delta x = \frac{1}{n}$, $a = 3$,

$b = 4$, and $f(x) = \sqrt{x}$, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{3+\frac{i}{n}} = \lim_{n\to\infty} R_n = \int_3^4 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big]_3^4$$

$$= \frac{2}{3}\left[8 - 3\sqrt{3}\right] = \underline{\underline{\frac{16}{3} - 2\sqrt{3}}}.$$

(a) Answer to Q.4(a) $\frac{1}{32}$

(b) Answer to Q.4(b) $\frac{16}{3} - 2\sqrt{3}$

(5) Find the equation of the tangent line to the curve defined by $x^3 + 2xy + y^3 = 13$ at the point $P(1,2)$. Give your final answer in the form $y = mx + b$.

Let f be the function whose graph describes the given curve near $P(1,2)$.

Thus $f(1) = 2$ and $\quad x^3 + 2x \cdot f(x) + [f(x)]^3 = 13$.

Differentiating with respect to x, we get

$$3x^2 + 2f(x) + 2x \cdot f'(x) + 3[f(x)]^2 \cdot f'(x) = 0.$$

Using $f(1) = 2$, we obtain at $x = 1$:

$$3 + 2 \cdot 2 + 2 \cdot f'(x) + 3 \cdot 2^2 \cdot f'(1) = 0, \text{ thus}$$

$$14 \cdot f'(1) = -7, \text{ so}$$

$$f'(1) = -\tfrac{1}{2}.$$

Hence the desired equation of the tangent line is

$$y - 2 = -\tfrac{1}{2}(x - 1), \text{ thus}$$

$$\underline{\underline{y = -\tfrac{1}{2}x + \tfrac{5}{2}}}.$$

Equation is: $y =$ $-\frac{1}{2}x + \frac{5}{2}$

(6) Find the absolute minimum value and the absolute maximum value of the function $f(x) = x + \frac{4}{x}$ on the interval $[1, 5]$.

The function f is differentiable on $(1,5)$ and $f'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2} = \frac{(x-2)(x+2)}{x^2}$,

which is zero if and only if $x=2$ or $x=-2$. Hence the only critical number of f in the interval $(1,5)$ is 2. We compute

x	1	2	5
$f(x)$	5	4	$\frac{29}{5}$

Since $4 < 5 < \frac{29}{5}$ and f is continuous on $[1,5]$, the closed interval method gives that f attains its absolute minimum on $[1,5]$ at 2 and its absolute maximum at 5

Absolute maximum value: $\frac{29}{5}$

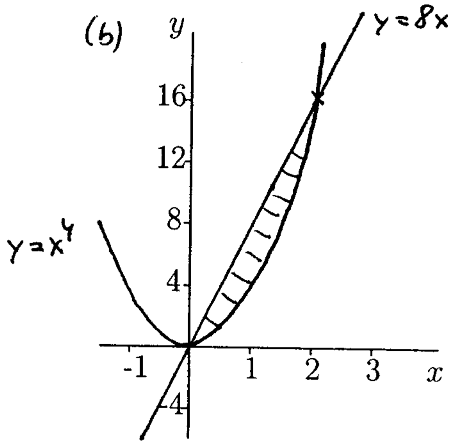
Absolute minimum value: 4

(7) (a) Determine the points where the two curves $y = 8x$ and $y = x^4$ meet.

(b) Sketch the region bounded by the two curves.

(c) Find the area of this region.

(b)



(a) The curves meet when $8x = x^4$, thus $0 = x^4 - 8x = x(x^3 - 8)$, which is equivalent to $x = 0$ or $x^3 = 8$, i.e. $x = 2$. Hence the intersection points are $(0,0)$ and $(2, 8\cdot 2) = (2, 16)$.

(c) The area A is

$$A = \int_0^2 [8x - x^4]\,dx = 4x^2 - \tfrac{1}{5}x^5 \Big]_0^2 = 16 - \frac{32}{5} = \underline{\underline{\frac{48}{5}}}.$$

Points of intersection: (0,0) and (2,16)

Area of the region $\frac{48}{5}$

(8) Evaluate the following integrals.

(a) $\int (2+x^3)^2\,dx.$ $= \int (4+4x^3+x^6)\,dx = 4x + x^4 + \frac{x^7}{7} + C.$

(b) $\int x \cdot \sqrt[3]{7-6x^2}\,dx.$ Substituting $u = 7-6x^2$, thus $du = -12x \cdot dx$, so $x\,dx = -\frac{1}{12}du$, we get

$$\int x \cdot \sqrt[3]{7-6x^2}\,dx = \int -\frac{1}{12}\sqrt[3]{u}\,du = -\frac{1}{12}\int u^{\frac{1}{3}}du = -\frac{1}{12}\cdot\frac{3}{4}u^{\frac{4}{3}} + C$$

$$= -\frac{1}{16}(7-6x^2)^{\frac{4}{3}} + C.$$

(c) $\int_0^{\pi/3} \frac{\sin(t)}{\cos^2(t)}\,dt.$ Substituting $u = \cos(t)$, thus $du = -\sin(t)\,dt$, we get using $\cos(0)=1$ and $\cos(\frac{\pi}{3}) = \frac{1}{2}$

$$\int_0^{\frac{\pi}{3}} \frac{\sin(t)}{\cos^2(t)}dt = \int_1^{\frac{1}{2}} \frac{-1}{u^2}du = \frac{1}{u}\Big]_1^{\frac{1}{2}} = 2-1 = 1.$$

(a) Answer to Q.8(a) $4x + x^4 + \frac{x^7}{7} + C$

(b) Answer to Q.8(b) $-\frac{1}{16}(7-6x^2)^{\frac{4}{3}} + C$

(c) Answer to Q.8(c) 1

Work two of the following three problems. Indicate the problem that is not to be graded by crossing through its number on the front of the exam.

(9) (a) State both parts of the Fundamental Theorem of Calculus. Use complete sentences.

(b) Consider the function f on $[1,\infty)$ defined by $f(x) = \int_1^x \sqrt{t^5 - 1}\, dt$. Argue that f is increasing.

(c) Find the derivative of the function $g(x) = \int_1^{x^3} \sqrt{t^5 - 1}\, dt$ on $(1,\infty)$.

(a) Let f be a continuous function on the interval $[a,b]$. Then the function g on $[a,b]$ defined by $g(x) = \int_a^x f(t)\,dt$ is continuous and differentiable on (a,b) with $g'(x) = f(x)$. Moreover, if F is any antiderivative of f on (a,b), then $\int_a^b f(x)\,dx = F(b) - F(a)$.

(b) By the Fundamental Theorem of Calculus, $f'(x) = \sqrt{x^5 - 1}$. Hence $f'(x) > 0$ if $x > 1$. It follows that f is increasing.

(c) Since $g(x) = f(x^3)$, the chain rule provides

$$g'(x) = f'(x^3) \cdot 3x^2 = 3x^2 \sqrt{(x^3)^5 - 1}$$

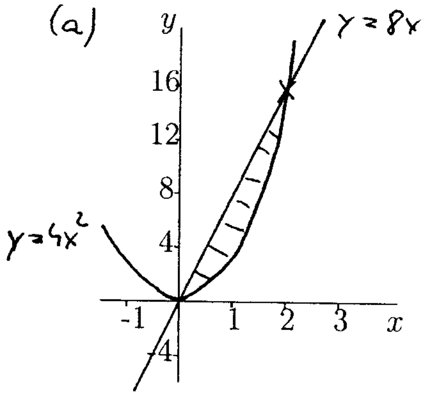
$$= 3x^2 \sqrt{x^{15} - 1}.$$

(10) Consider the region bounded by the curves $y = 8x$ and $y = 4x^2$.

(a) Sketch the region.

(b) Find the volume of the solid obtained by revolving the region about the y-axis.

(a)



(b) The two curves meet when $8x = 4x^2$, i.e. $0 = 4x^2 - 8x = 4x(x-2)$, which is equivalent to $x=0$ or $x=2$. Hence the intersection points are $(0,0)$ and $(2,16)$.

If $x \geq 0$, then $y = 8x$ is equivalent to $x = \frac{1}{8}y$

and $y = 4x^2$ is equivalent to $x = \frac{1}{2}\sqrt{y}$.

Hence the desired volume is

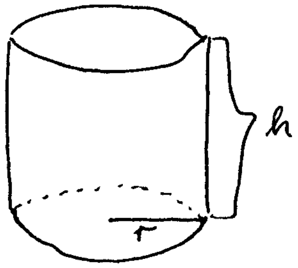
$$V = \pi \int_0^{16} \left[\left(\tfrac{1}{2}\sqrt{y}\right)^2 - \left(\tfrac{1}{8}y\right)^2 \right] dy = \pi \int_0^{16} \left[\frac{y}{4} - \frac{y^2}{64} \right] dy$$

$$= \pi \left[\frac{y^2}{8} - \frac{y^3}{3 \cdot 16 \cdot 4} \right]_0^{16}$$

$$= \pi \left[32 - \frac{64}{3} \right] = \underline{\underline{\frac{32}{3}\pi}}.$$

(b) Volume: $\frac{32}{3}\pi$

(11) A company wants to produce cylindrical beer glasses that can hold 0.5 liter beer. It costs 1 cent per square centimeter to manufacture the side of the glass and $\frac{4}{\pi}$ cents per square centimeter to manufacture its bottom. Find the dimensions (in centimeters) and the cost of the cheapest such glass.
(Note that 1 liter equals 1000 cm^3.)



Let r be the radius and h be the height of the glass. Then the area of the bottom is πr^2 and the area of the side is $2\pi r \cdot h$ (= circumference × height).

Thus, the cost to produce the bottom is $\frac{4}{\pi} \cdot \pi r^2 = 4r^2$ and for the side is $2\pi r h$ cents if we measure r and h in cm. Hence the total costs are $C = 2\pi r h + 4r^2$.

The volume of the glass is $\pi r^2 h$. Since 0.5 liter equal 500 cm^3, we get $500 = \pi r^2 h$ or $h = \frac{500}{\pi r^2}$. Using this to eliminate h from the costs, we get $C(r) = 2\pi r \cdot \frac{500}{\pi r^2} + 4r^2$

$$= \frac{1000}{r} + 4r^2,$$ which we want to minimize on $(0, \infty)$.

Using $C'(r) = -\frac{1000}{r^2} + 8r$, we get $C'(r) = 0$ if and only if $\frac{1000}{r^2} = 8r$, so $125 = r^3$, thus $r = 5$ is the only critical number of C in $(0, \infty)$

Moreover, we get

interval	$(0,5)$	$(5,\infty)$
sign of $C'(x)$	$-$	$+$
I/D	decreasing	increasing

Hence the first derivative test shows that C attains its absolute minimum at $\underline{r = 5}$. Then its height is $h = \frac{500}{\pi \cdot 5^2} = \underline{\frac{20}{\pi}}$ and the cost is $C(5) = \frac{1000}{5} + 4 \cdot 5^2 = \underline{300}$.

Height: $\frac{20}{\pi}$ cm Radius of the base: 5 cm

Cost: 300 cents

Extra Credit Problem.

Determine if the following statements are true or false. No justification is required. Each correct answer is worth 2 points. Each false answer leads to a deduction of 1 point. However, your total score for this problem will not be below zero.

(a) Suppose the function f is continuous on $[1,5]$ and differentiable on $(1,5)$. If $f(1) = 5$ and $f(5) = 20$, then $f'(t) \geq 4$ for at least one t in $(1,5)$.

Answer: True ☐ **False** ☑

(b) Suppose the function f is differentiable on the set of real numbers. Then f is continuous.

Answer: True ☑ **False** ☐

(c) The function $f(x) = \int_1^x |t| \, dt$ is differentiable.

Answer: True ☑ **False** ☐

(d) If $f'(a) = 0$, then f has a local maximum or a local minimum at a.

Answer: True ☐ **False** ☑

(e) If f is continuous function on the open interval $(1,10)$, then f has an absolute maximum at some point c in $(1,10)$.

Answer: True ☐ **False** ☑