Worksheet # 9: Limits at infinity and Intermediate Value Theorem

1. (a) Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.
   (b) Explain the difference between "$\lim_{x\to-3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".

2. Evaluate the following limits, or explain why the limit does not exist:
   (a) $\lim_{x\to\infty} \dfrac{3x^2 - 7x}{x - 8}$
   (b) $\lim_{x\to\infty} \dfrac{2x^2 - 6}{x^4 - 8x + 9}$
   (c) $\lim_{x\to-\infty} \dfrac{x}{x^6 - 4x^2}$
   (d) $\lim_{x\to-\infty} 3$
   (e) $\lim_{x\to\pm\infty} \dfrac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$
   (f) $\lim_{x\to-\infty} \dfrac{\sqrt{x^{10} + 2x}}{x^5}$

3. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \left(\dfrac{x^2}{x+1} - \dfrac{x^2}{x-1}\right)$.

4. Sketch a graph with all of the following properties:
   - $\lim_{t\to\infty} f(t) = 2$
   - $\lim_{t\to-\infty} f(t) = 0$
   - $\lim_{t\to0^+} f(t) = \infty$
   - $\lim_{t\to0^-} f(t) = -\infty$
   - $\lim_{t\to4} f(t) = 3$
   - $f(4) = 6$

5. Find the following limits;
   (a) $\lim_{x\to\infty} \dfrac{3x + 2\sqrt{x}}{1 - x}$
   (b) $\lim_{x\to-\infty} \dfrac{2x - 5}{|3x + 2|}$
   (c) $\lim_{x\to\infty} \dfrac{5x^2 + \sin x}{3x^2 + \cos x}$

6. (a) State the Intermediate Value Theorem.
   (b) Show that $f(x) = x^3 + x - 1$ has a zero in the interval $[0, 1]$.

7. Use the Intermediate Value Theorem to find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

8. Show that there is some a with $0 < a < 2$ such that $a^2 + \cos(\pi a) = 4$.

9. Show that the equation $\ln(x) = e^{-x}$ has a solution between 1 and 2.

10. Let $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$ be a piecewise function.
    Although $f(-1) = 0$ and $f(1) = 1$, $f(x) \neq 1/2$ for all x in its domain. Why doesn't this contradict to the Intermediate Value Theorem?

11. Prove that $x^4 = -1$ has no solution.