Worksheets for MA 113

http://www.math.uky.edu/~ma113/

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Sunday 17th August, 2014

Worksheet # 1: Functions and inverse functions

1. Give the domain and ranges of the following functions.

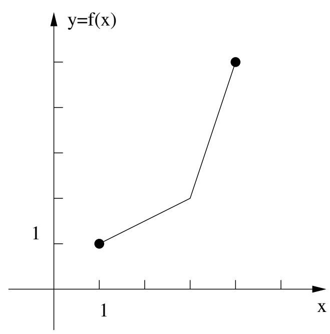
   (a) $f(x) = \frac{x+1}{x^2+x-2}$

   (b) $g(t) = \frac{1}{\sqrt{t^2-1}}$

2. If $f(x) = 5x + 7$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. Are the functions $f \circ g$ and $g \circ f$ the same function?

3. Let $f(x) = 2 + \frac{1}{x+3}$. Determine the inverse function of f, f^{-1}. Give the domain and range of f and the inverse function f^{-1}. Verify that $f \circ f^{-1}(x) = x$.

4. Consider the function whose graph appears below.



   (a) Find $f(3)$, $f^{-1}(2)$ and $f^{-1}(f(2))$.

   (b) Give the domain and range of f and of f^{-1}.

   (c) Sketch the graph of f^{-1}.

5. Let $f(x) = x^2 + 2x + 5$. Find the largest value of a so that f is one to one on the interval $(-\infty, a]$. Let g be the function f with the domain $(-\infty, a]$. Find the inverse function g^{-1}. Give the domain and range of g^{-1}.

6. True or False:

   (a) Every function has an inverse.

   (b) If $f \circ g(x) = x$ for all x in the domain of g, then f is the inverse of g.

   (c) If $f \circ g(x) = x$ for all x in the domain of g and $g \circ f(x) = x$ for all x in the domain of f, then f is the inverse of g.

   (d) If $f(x) = 1/(x + 2)^3$ and g is the inverse function of f, then $g(x) = (x + 2)^3$.

   (e) The function $f(x) = \sin(x)$ is one to one.

   (f) The function $f(x) = 1/(x + 2)^3$ is one to one.

7. Find the slope, x-intercept, and y-intercept of the line $3x - 2y = 4$.

8. Let f be a linear function with slope m with $m \neq 0$. What is the slope of the inverse function f^{-1}.

9. A ball is thrown in the air from ground level. The height of the ball in meters at time t seconds is given by the function $h(t) = -4.9t^2 + 30t$. At what time does the ball hit the ground (be sure to use the proper units)?

10. We form a box by removing squares of side length x centimeters from the four corners of a rectangle of width 100 cm and length 150 cm and then folding up the flaps between the squares that were removed. a) Write a function which gives the volume of the box as a function of x. b) Give the domain for this function.

Worksheet # 2: Review of Trigonometry

1. Convert the angle $\pi/12$ to degrees and the angle $900°$ to radians. Give exact answers.

2. Suppose that $\sin(\theta) = 5/13$ and $\cos(\theta) = -12/13$. Find the values of $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, and $\sec(\theta)$. Find the value of $\tan(2\theta)$.

3. If $\pi/2 \leq \theta \leq 3\pi/2$ and $\tan\theta = 4/3$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

4. Find all solutions of the equations a) $\sin(x) = -\sqrt{3}/2$, b) $\tan(x) = 1$.

5. A ladder that is 6 meters long leans against a wall so that the bottom of the ladder is 2 meters from the base of the wall. Make a sketch illustrating the given information and answer the following questions.

   How high on the wall is the top of the ladder located? What angle does the top of the ladder form with the wall?

6. Let O be the center of a circle whose circumference is 48 centimeters. Let P and Q be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments OQ and OP. Express your answer in radians.

   Find the distance between P and Q.

7. The center of a clock is located at the origin so that 12 lies on the positive y-axis and the 3 lies on the positive x-axis. The minute hand is 10 units long and the hour hand is 7 units. Find the coordinates of the tips of the minute hand and hour hand at 9:50 am on Newton's birthday.

8. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

   (a) $\sqrt{3}\cos(x) + 2\tan(x)\cos^2(x) = 0$
   (b) $3\cot^2(x) = 1$
   (c) $2\cos(x) + \sin(2x) = 0$

9. A function is said to be periodic with period T if $f(x) = f(x+T)$ for any x. Find the smallest, positive period of the following trigonometric functions. Assume that ω is positive.

   (a) $|\sin t|$
   (b) $\sin(3t)$.
   (c) $\sin(\omega t) + \cos(\omega t)$.
   (d) $\tan^2(\omega t)$.

10. Find a quadratic function $p(x)$ so that the graph p has x-intercepts at $x = 2$ and $x = 5$ and the y-intercept is $y = -2$.

11. Find the exact values of the following expressions. Do not use a calculator.

    (a) $\tan^{-1}(1)$
    (b) $\tan(\tan^{-1}(10))$
    (c) $\sin^{-1}(\sin(7\pi/3))$
    (d) $\tan(\sin^{-1}(0.8))$

12. Give a simple expression for $\sin(\cos^{-1}(x))$.

13. Let f be the function with domain $[\pi/2, 3\pi/2]$ with $f(x) = \sin(x)$ for x in $[\pi/2, 3\pi/2]$. Since f is one to one, we may let g be the inverse function of f. Give the domain and range of g. Find $g(1/2)$.

Worksheet # 3: The Exponential Function and the Logarithm

1. (a) Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ and give the domains and range of each function.
   (b) Determine if each function is one-to-one. Determine if each function is increasing or decreasing.
   (c) Graph the inverse function to f. Give the domain and range of the inverse function.

2. (a) Solve $10^{2x+1} = 100$.
   (b) Solve $2^{(x^2)} = 16$.
   (c) Solve $2^x = 4^{x+2}$.
   (d) Find $\log_2(8)$.
   (e) Find $\ln(e^2)$.
   (f) Solve $e^{3x} = 3$.
   (g) Solve $4^x = e$.
   (h) Solve $\ln(x+1) + \ln(x-1) = \ln(3)$. Be sure to check your answer.

3. Evaluate the expressions $4^{(3^2)}$ and $(4^3)^2$. Are they equal?

4. Suppose a and b are positive real numbers and $\ln(ab) = 3$ and $\ln(ab^2) = 5$. Find $\ln(a)$, $\ln(b)$, and $\ln(a^3/\sqrt{b})$.

5. Consider the function $f(x) = 1 + \ln(x)$. Determine the inverse function of f. Give the domain and range of f and of the inverse function f^{-1}.

6. Suppose that a population doubles every two hours. If we have one hundred critters at 12 noon, how many will there be after 1 hour? after 2 hours? How many were there at 11am? Give a formula for the number of critters at t hours after 12 noon.

7. Let f be the function $f(x) = 4^x$. Write the function f in the form $f(x) = e^{kx}$.

8. Let f be the function $f(x) = 5 \cdot 3^x$. Write the function in the form Ae^{kx}.

9. Suppose that f is a function of the form $f(x) = Ae^{kx}$. If $f(2) = 20$ and $f(5) = 10$, will we have $k > 0$ or $k < 0$? Find A and k so that $f(2) = 20$ and $f(5) = 10$.

10. Let f be the function given by $f(x) = \tan(x)$ with x in the interval $(\pi/2, 3\pi/2)$. Find $f^{-1}(0)$ and $f^{-1}(\sqrt{3})$.

    Sketch the graphs of f and f^{-1}. Give a formula which expresses f^{-1} in terms of the function $\tan^{-1}$ (or arctan).

Worksheet # 4: Limits: A Numerical and Graphical Approach, the Limit Laws

1. Comprehension check:

   (a) In words, describe what "$\lim_{x\to a} f(x) = L$" means.

   (b) In words, what does "$\lim_{x\to a} f(x) = \infty$" mean?

   (c) Suppose $\lim_{x\to 1} f(x) = 2$. Does $f(1) = 2$?

   (d) Suppose $f(1) = 2$. Does $\lim_{x\to 1} f(x) = 2$?

2. Let $p(t)$ denote the distance (in meters) to the right of the origin of a particle at time t minutes after noon. If $p(t) = p(t) = t^3 - 45t$, give the average velocity on the intervals $[2, 2.1]$ and $[2, 2.01]$. Use this information to guess a value for the instantaneous velocity of particle at 12:02pm.

3. Consider the circle $x^2 + y^2 = 25$ and verify that the point $(-3, 4)$ lies on the circle. Find the slope of the secant line that passes through the points with x-coordinates -3 and -3.1. Find the slope of the secant line that passes through the points with x-coordinates -3 and -2.99.

   Use this information to guess the slope of the tangent line to the circle at $(-3, 4)$. Write the equation of the tangent line and use a graphing calculator to verify that you have found the tangent line.

4. Compute the value of the following functions near the given $x-$value. Use this information to guess the value of the limit of the function (if it exists) as x approaches the given value.

   (a) $f(x) = \frac{4x^2-9}{2x-3}$, $x = \frac{3}{2}$

   (b) $f(x) = \frac{x}{|x|}$, $x = 0$

   (c) $f(x) = \frac{\sin(2x)}{x}$, $x = 0$

   (d) $f(x) = \sin(\pi/x)$, $x = 0$

5. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x-1 & \text{if } 0 < x \text{ and } x \ne 2 \\ -3 & \text{if } x = 2 \end{cases}$.

   (a) Sketch the graph of f.

   (b) Compute the following:

      i. $\lim_{x\to 0^-} f(x)$

      ii. $\lim_{x\to 0^+} f(x)$

      iii. $\lim_{x\to 0} f(x)$

      iv. $f(0)$

      v. $\lim_{x\to 2^-} f(x)$

      vi. $\lim_{x\to 2^+} f(x)$

      vii. $\lim_{x\to 2} f(x)$

      viii. $f(2)$

6. Compute the following limits or explain why they fail to exist:

   (a) $\displaystyle\lim_{x\to -3^+} \frac{x+2}{x+3}$

   (b) $\displaystyle\lim_{x\to -3^-} \frac{x+2}{x+3}$

   (c) $\displaystyle\lim_{x\to -3} \frac{x+2}{x+3}$

   (d) $\displaystyle\lim_{x\to 0^-} \frac{1}{x^3}$

7. Given $\lim_{x\to 2} f(x) = 5$ and $\lim_{x\to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit. Note when working through a limit problem that your answers should be a chain of true equalities. Make sure to keep the $\lim_{x\to a}$ operator until the very last step.

(a) $\lim_{x\to 2}(2f(x) - g(x))$

(b) $\lim_{x\to 2}(f(x)g(2))$

(c) $\lim_{x\to 2}\dfrac{f(x)g(x)}{x}$

(d) $\lim_{x\to 2} f(x)^2 + x \cdot g(x)^2$

(e) $\lim_{x\to 2}[f(x)]^{\frac{3}{2}}$

(f) $\lim_{x\to 2}\dfrac{f(x)-5}{g(x)-2}$

8. Let $f(x) = \begin{cases} 2x+2 & \text{if } x > -2 \\ a & \text{if } x = -2 \\ kx & \text{if } x < -2 \end{cases}$. Find k and a so that $\lim_{x\to -2} f(x) = f(-2)$.

9. In the theory of relativity, the mass of a particle with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

Worksheet # 5: Continuity

1. Comprehension check:

   (a) Define what it means for $f(x)$ to be continuous at the point $x = a$. What does it mean if $f(x)$ is continuous on the interval $[a, b]$? What does it mean to say $f(x)$ is continuous?

   (b) There are three distinct ways in which a function will fail to be continuous at a point $x = a$. Describe the three types of discontinuity. Provide a sketch and an example of each type.

   (c) True or false? Every function is continuous on its domain.

   (d) True or false? The sum, difference, and product of continuous functions are all continuous.

   (e) If $f(x)$ is continuous at $x = a$, what can you say about $\lim_{x \to a^+} f(x)$?

2. Using the definition of continuity and properties of limits, show that the following functions are continuous at the given point a.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \dfrac{x^2 + 3x + 1}{x + 3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2 - 9}$, $a = 4$

3. Give the intervals of continuity for the following functions.

   (a) $f(x) = \dfrac{x + 1}{x^2 + 4x + 3}$

   (b) $f(x) = \dfrac{x}{x^2 + 1}$

   (c) $f(x) = \sqrt{2x - 3} + x^2$

   (d) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ x + 1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \ge 2 \end{cases}$

4. If $\lim_{h \to 3}(hf(h)) = 15$, can you find $\lim_{h \to 3} f(h)$? Use the limit laws to justify your answer.

5. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.

   (a) Find all numbers c such that $\lim_{x \to 1} f(x)$ exists.

   (b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

6. Find parameters a and b so that $f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \le -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \le x \end{cases}$ is continuous.

7. Suppose that $f(x)$ and $g(x)$ are continuous functions where $f(2) = 5$ and $g(6) = 1$. Compute the following:

   (a) $\lim_{x \to 2} \dfrac{[f(x)]^2 + x}{3x + 2}$.

   (b) $\lim_{x \to 6} \dfrac{g(x) + 4x}{f\left(\frac{x}{3}\right) - g(x)}$

8. Suppose that: $f(x) = \begin{cases} \frac{x-6}{|x-6|} & \text{for } x \ne 6, \\ 1 & \text{for } x = 6 \end{cases}$

   Determine the points at which the function $f(x)$ is discontinuous and state the type of discontinuity.

Worksheet # 6: Algebraic Evaluation of Limits, Trigonometric Limits

1. For each limit, evaluate the limit or or explain why it does not exist. Use the limit rules to justify each step. It is good practice to sketch a graph to check your answers.

   (a) $\lim_{x\to 2} \dfrac{x+2}{x^2-4}$

   (b) $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

   (c) $\lim_{x\to 2} \dfrac{x-2}{x^2+4}$

   (d) $\lim_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{3}{x^2-x-2}\right)$

   (e) $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

   (f) $\lim_{x\to 0} \dfrac{(2+x)^3-8}{x}$

2. Let $f(x) = \sqrt{x}$

   (a) Let $g(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ and find $g(x)$.

   (b) What is the geometric meaning of $g(4)$?

   (c) What is the domain of $g(x)$?

3. Let $f(x) = 1+x^2\sin(1/x)$ for $x \neq 0$. Find two simpler functions g and h so that we can use the squeeze theorem to show $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$. Give the common value of the limits.

   Use your calculator to produce a graph that illustrates that the squeeze theorem applies.

4. Evaluate the limits

   (a) $\lim_{t\to 0} \dfrac{\sin(2t)}{t}$

   (b) $\lim_{t\to 0} \dfrac{\tan(2t)}{t}$

   (c) $\lim_{t\to 0} \dfrac{\cos(t)\tan(2t)}{t}$

   (d) $\lim_{t\to 0} \csc(3t)\tan(2t)$

   (e) $\lim_{t\to 0} \dfrac{1-\cos(2t)}{t}$

   (f) $\lim_{t\to 0} \dfrac{1-\cos(2t)}{\sin^2(2t)}$ **Hint**: Multiply by $\frac{1+\cos(2t)}{1+\cos(2t)}$.

   (g) $\lim_{t\to \pi/2} \dfrac{1-\cos(t)}{t}$

   The following identity may be useful for the next problems.

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{1}$$

5. Use equation (1), to simplify the limit

$$\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

6. Evaluate the following limits

   (a) $\lim_{t\to 0} \dfrac{t^2}{\sin t}$

   (b) $\lim_{t\to 0} \dfrac{\cos(5t)-\cos^2(5t)}{t}$

   (c) $\lim_{x\to 0} \dfrac{\tan(11x)}{5x}$

   (d) $\lim_{x\to 0} \dfrac{\cos(2x)-1}{\cos(x)-1}$ **Hint:** Use equation (1)

   (e) $\lim_{x\to 0} \dfrac{1-\cos(3x)}{x^2}$ **Hint:** Multiply and divide by $1+\cos(3x)$

   (f) $\lim_{x\to 0} \dfrac{\cos x-\cos(4x)}{x^2}$ **Hint:** Use equation (1) to rewrite $\cos(4x)$ as $\cos(x+3x)$

Worksheet # 7: The Intermediate Value Theorem

1. (a) State the Intermediate Value Theorem.

   (b) Show that $f(x) = x^3 + x - 1$ has a zero in the interval $[0, 1]$.

2. Let $f(x) = e^x/(e^x - 2)$. Show that $f(0) < 1 < f(\ln(4))$.

   Can you use the intermediate value theorem to conclude that there is a solution of $f(x) = 1$?

   Can you find a solution to $f(x) = 1$?

3. Use the Intermediate Value Theorem to find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

4. Show that there is some a with $0 < a < 2$ such that $a^2 + \cos(\pi a) = 4$.

5. Show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$.

   Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.

   Continue to find an interval of length $1/8$ which contains a solution of the equation $xe^x = 2$.

6. Find an interval for which the equation $\ln(x) = e^{-x}$ has a solution. Use the intermediate value theorem to check your answer.

7. Let $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ be a piecewise function.

   Although $f(-1) = 0$ and $f(1) = 1$, $f(x) \neq 1/2$ for all x in its domain. Why doesn't this contradict to the Intermediate Value Theorem?

8. Prove that $x^4 = -1$ has no solution.

9. Determine if the following are true or false.

   (a) If f is continuous on $[-1, 1]$, $f(-1) = -2$ and $f(1) = 2$, then $f(0) = 0$.

   (b) If f is continuous on $[-1, 1]$, $f(-1) = -2$ and the equation $f(x) = 1$ has no solution, then there is no solution to $f(x) = 0$.

   (c) If f is continuous on $[-1, 1]$, $f(-1) = 1$, and the equation $f(x) = 1$ has no solution, then there is no solution to $f(x) = 2$.

   (d) If f is a function with domain $[-1, 1]$, $f(-1) = -2$, $f(1) = 2$, and for each real number y, we can find a solution to the equation $f(x) = y$, then f is continuous.

10. (Review) A particle moves along a line and its position after time t seconds is $p(t) = 3t^3 + 2t$ meters to the right of the origin. Find the instantaneous velocity of the particle at $t = 2$.

Worksheet # 8: Review for Exam I

1. Find all real numbers of the constant a and b for which the function $f(x) = ax + b$ satisfies:
   (a) $f \circ f(x) = f(x)$ for all x.
   (b) $f \circ f(x) = x$ for all x.

2. Simplify the following expressions.
   (a) $\log_5 125$
   (b) $(\log_4 16)(\log_4 2)$
   (c) $\log_{15} 75 + \log_{15} 3$
   (d) $\log_x(x(\log_y y^x))$
   (e) $\log_\pi(1 - \cos x) + \log_\pi(1 + \cos x) - 2\log_\pi \sin x$

3. Suppose that $\tan(x) = 3/4$ and $-\pi < x < 0$. Find $\cos(x)$, $\sin(x)$, and $\sin(2x)$.

4. (a) Solve the equation $3^{2x+5} = 4$ for x. Show each step in the computation.
   (b) Express the quantity $\log_2(x^3 - 2) + \frac{1}{3}\log_2(x) - \log_2(5x)$ as a single logarithm. For which x is the resulting identity valid?

5. Calculate the following limits using the limit laws. Carefully show your work and use only one limit law per step.
   (a) $\lim_{x \to 0}(2x - 1)$
   (b) $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$

6. Determine if the following limits exist and calculate the limit when possible.
   (a) $\lim_{x \to 1} \frac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
   (b) $\lim_{x \to 2} \frac{x - 2}{\frac{1}{x} - \frac{1}{2}}$
   (c) $\lim_{x \to 2} \frac{x^2}{x - 2}$
   (d) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$
   (e) $\lim_{x \to 2} \frac{x + 1}{x - 2}$

7. Suppose that the height of an object at time t is $h(t) = 5t^2 + 40t$.
   (a) Find the average velocity of the object on the interval $[3, 3.1]$.
   (b) Find the average velocity of the object on the interval $[a, a + h]$.
   (c) Find the instantaneous velocity of the object time a.

8. Use the fact that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ to find $\lim_{x \to 0} \frac{x}{\sin(3x)}$.

9. (a) State the Squeeze Theorem.
   (b) Use the Squeeze Theorem to find the limit $\lim_{x \to 0} x \sin \frac{1}{x^2}$

10. Use the Squeeze Theorem to find $\lim_{x \to \frac{\pi}{2}} \cos x \cos(\tan x)$

11. If $f(x) = \dfrac{|x-3|}{x^2-x-6}$, find $\lim\limits_{x\to 3^+} f(x)$, $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3} f(x)$.

12. (a) State the definition of the continuity of a function $f(x)$ at $x = a$.

    (b) Find the constant a so that the function is continuous on the entire real line.

$$f(x) = \begin{cases} \dfrac{x^2-a^2}{x-a} & \text{if } x \neq a \\ 8 & \text{if } x = a \end{cases}$$

13. Complete the following statements:

    (a) A function $f(x)$ passes the horizontal line test, if the function f is ....

    (b) If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \quad \text{provided ....}$$

    (c) $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = L$ if and only if ....

    (d) Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ be a piecewise function.
    The function $g(x)$ is NOT continuous at $x = 2$ since ....

    (e) Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$ be a piecewise function.
    The function $f(x)$ is NOT continuous at $x = 0$ since ....

14. If $g(x) = x^2 + 5^x - 3$, use the Intermediate Value Theorem to show that there is a number a such that $g(a) = 10$.

Worksheet # 9: The Derivative

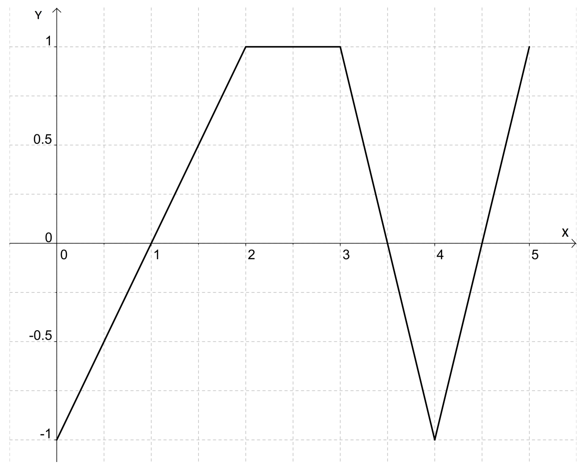
1. Comprehension check:

   (a) What is the definition of the derivative $f'(a)$ at a point a?

   (b) What is the geometric meaning of the derivative $f'(a)$ at a point a?

   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$?

2. Consider the graph below of the function $f(x)$ on the interval $[0, 5]$.



   (a) For which x values would the derivative $f'(x)$ not be defined?

   (b) Sketch the graph of the derivative function f'.

3. Find a function f and a number a so that the following limit represents a derivative $f'(a)$.

$$\lim_{h \to 0} \frac{(4+h)^3 - 64}{h}$$

4. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

5. Find the specified derivative for each of the following.

   (a) If $f(x) = 1/x$, find $f'(2)$.

   (b) If $g(x) = \sqrt{x}$, find $g'(2)$.

   (c) If $h(x) = x^2$, find $h'(s)$.

   (d) If $f(x) = x^3$, find $f'(-2)$.

   (e) If $g(x) = 1/(2-x)$, find $g'(t)$.

6. Let

$$g(t) = \begin{cases} at^2 + bt + c & \text{if } t \leq 0 \\ t^2 + 1 & \text{if } t > 0 \end{cases}.$$

   Find all values of a, b, and c so that g is differentiable at $t = 0$.

7. Let $f(x) = e^x$ and estimate the derivative $f'(0)$ by considering difference quotients $(f(h) - f(0))/h$ for small values of h.

8. Suppose that $f'(0)$ exists. Does the limit

$$\lim_{h\to 0} \frac{f(h) - f(-h)}{h}$$

exist? Can you express the limit in terms of $f'(0)$?

9. Can you find a function f so that

$$\lim_{h\to 0} \frac{f(h) - f(-h)}{h}$$

exists, but f is not differentiable at 0?

10. Find A and B so that the limit

$$\lim_{x\to 1} \frac{x^2 + 2x - (Ax + B)}{(x-2)^2}$$

is finite. Give the value of the limit.

Worksheet # 10: The Derivative as Function, Product and Quotient Rules

1. Comprehension check:
   (a) True or false: If $f'(x) = g'(x)$ for all x, then $f = g$?
   (b) True or false: If $f(x) = g(x)$ for all x, then $f' = g'$?
   (c) True or false: $(f + g)' = f' + g'$
   (d) True or false: $(fg)' = f'g'$
   (e) How is the number e defined?
   (f) Are differentiable functions also continuous? Are continuous functions also differentiable?
2. Show by way of example that, in general,
$$\frac{d}{dx}(f \cdot g) \neq \frac{df}{dx} \cdot \frac{dg}{dx}$$
   and
$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}.$$
3. Calculate the derivatives of the following functions in the two ways that are described.
   (a) $f(r) = r^3/3$
      i. using the constant multiple rule and the power rule
      ii. using the quotient rule and the power rule

   Which method should we prefer?
   (b) $f(x) = x^5$
      i. using the power rule
      ii. using the product rule by considering the function as $f(x) = x^2 \cdot x^3$
   (c) $g(x) = (x^2 + 1)(x^4 - 1)$
      i. first multiply out the factors and then use the power rule
      ii. by using the product rule
4. State the quotient and product rule and be sure to include all necessary hypotheses.
5. Compute the first derivative of each of the following:
   (a) $f(x) = (3x^2 + x)e^x$
   (b) $f(x) = \dfrac{\sqrt{x}}{x - 1}$
   (c) $f(x) = \dfrac{e^x}{2x^3}$
   (d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x - 1}{\sqrt{x}}\right)$
   (e) $f(x) = \dfrac{2x}{4 + x^2}$
   (f) $f(x) = \dfrac{ax + b}{cx + d}$
   (g) $f(x) = \dfrac{(x^2 + 1)(x^3 + 2)}{x^5}$
   (h) $f(x) = (x - 3)(2x + 1)(x + 5)$
6. Let $f(x) = (3x - 1)e^x$. For which x is the slope of the tangent line to f positive? Negative? Zero?

7. Find an equation of the tangent line to the given curve at the specified point. Sketch the curve and the tangent line to check your answer.

   (a) $y = x^2 + \dfrac{e^x}{x^2+1}$ at the point $x = 3$.

   (b) $y = 2xe^x$ at the point $x = 0$.

8. Suppose that $f(2) = 3,\ g(2) = 2,\ f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

   (a) $h(x) = 5f(x) + 2g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   (d) $h(x) = \dfrac{g(x)}{1+f(x)}$

Worksheet # 11: Rates of Change

1. Given that $G(t) = 4t^2 - 3t + 42$ find the instantaneous rate of change when $t = 3$.

2. A particle moves along a line so that its position at time t is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

3. An object is thrown upward so that its height at time t seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time t.

4. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s^2.

5. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s^2.

6. Suppose that height of a triangle is equal to its base b. Find the instantaneous rate of change in the area respect to the base b when the base is 7.

7. The cost in dollars of producing x bicycles is $C(x) = 4000 + 210x - x^2/1000$.

   (a) Explain why $C'(40)$ is a good approximation for the cost of the 41st bicycle.

   (b) How can you use the values of $C(40)$ and $C'(40)$ to approximate the cost of 42 bicycles?

   (c) Explain why the model for $C(x)$ is not a good model for cost. What happens if x is very large?

8. Suppose that f is a function with $f(7) = 34$ and $f'(7) = 4$. Explain why we can use 42 as an approximation for $f(9)$. What is an approximate value for $f(6)$?

9. (Review) Suppose that the tangent line to the graph of f at $(2, f(2))$ is $y = 3x + 4$. Find the tangent line to the graph of $xf(x)$.

10. (Review) Differentiate the following functions.

    (a) $f(x) = (x + 1)(x - 2)$

    (b) $g(x) = \frac{x^2-1}{x^2+1}$

    (c) $h(x) = x \sin(x)$

Worksheet # 12: Higher Derivatives and Trigonometric Functions

1. Calculate the indicated derivative:

   (a) $f^{(4)}(1)$, $\qquad f(x) = x^4$
   (b) $g^{(3)}(5)$, $\qquad g(x) = 2x^2 - x + 4$
   (c) $h^{(3)}(t)$, $\qquad h(t) = 4e^t - t^3$
   (d) $s^{(2)}(w)$, $\qquad s(w) = \sqrt{w}e^w$

2. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the n-th derivative of f.

3. Let $f(t) = t + 2\cos(t)$.

   (a) Find all values of t where the tangent line to f at the point $(t, f(t))$ is horizontal.
   (b) What are the largest and smallest values for the slope of a tangent line to the graph of f?

4. Differentiate each of the following functions:

   (a) $f(t) = \cos(t)$
   (b) $g(u) = \dfrac{1}{\cos(u)}$
   (c) $r(\theta) = \theta^3 \sin(\theta)$
   (d) $s(t) = \tan(t) + \csc(t)$
   (e) $h(x) = \sin(x)\csc(x)$
   (f) $f(x) = x^2 \sin(x)$
   (g) $g(x) = \sec(x) + \cot(x)$

5. Calculate the first five derivatives of $f(x) = \sin(x)$. Then determine $f^{(8)}$ and $f^{(37)}$

6. Calculate the first 5 derivatives of $f(x) = 1/x$. Can you guess a formula for the nth derivative, $f^{(n)}$?

7. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

   (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.
   (b) Is the particle moving towards or away from the origin at π seconds? Explain.
   (c) Now, find the velocity of the particle at time $t = \dfrac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?
   (d) Is the particle speeding up at $\frac{\pi}{2}$ seconds?

8. Find an equation of the tangent line at the point specified:

   (a) $y = x^3 + \cos(x)$, $\qquad x = 0$
   (b) $y = \csc(x) - \cot(x)$, $\qquad x = \frac{\pi}{4}$
   (c) $y = e^\theta \sec(\theta)$, $\qquad \theta = \frac{\pi}{4}$

9. Comprehension check for derivatives of trigonometric functions:

   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.
   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \dfrac{2}{3}$, then the hypotenuse of the triangle must have length 3.

Worksheet # 13: Chain Rule

1. (a) Carefully state the chain rule using complete sentences.

   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \dfrac{1}{4}$, and $g'(2) = 2$. Find each of the following:

   i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.

   ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

2. Suppose that $k(x) = \sqrt{\sin^2(x) + 4}$. Find three functions f, g, and h so that $k(x) = f(g(h(x)))$.

3. Given the following functions: $f(x) = \sec(x)$, and $g(x) = x^3 - 2x + 1$. Find:

   (a) $f(g(x))$

   (b) $f'(x)$

   (c) $g'(x)$

   (d) $f'(g(x))$

   (e) $(f \circ g)'(x)$

4. Differentiate each of the following and simplify your answer.

   (a) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$

   (b) $g(t) = \tan(\sin(t))$

   (c) $h(u) = \sec^2(u) + \tan^2(u)$

   (d) $f(x) = xe^{(3x^2+x)}$

   (e) $g(x) = \sin(\sin(\sin(x)))$

5. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 e^{3x}$, $x = 2$

   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

6. Compute the derivative of $\frac{x}{x^2+1}$ in two ways:

   (a) Using the quotient rule.

   (b) Rewrite the function $\frac{x}{x^2+1} = x(x^2+1)^{-1}$ and use the product and chain rule.

   Check that both answers give the same result.

7. If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

8. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

   Find: $h'(-1), h'(3)$ and $k'(2)$.

9. Find all x values so that $f(x) = 2\sin(x) + \sin^2(x)$ has a horizontal tangent at x.

10. Suppose that at the instant when the radius of a circle of a circle is 5 cm, the radius is decreasing at a rate of 3 cm/sec. Find the rate of change of the area of the circle when the the radius is 5 cm.

11. Differentiate both sides of the double-angle formula for the cosine function, $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar identity?

Worksheet # 14: Implicit Differentiation and Inverse Functions

1. Find the derivative of y with respect to x:

   (a) $\ln(xy) = \cos(y^4)$.

   (b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.

   (c) $\sin(xy) = \ln\left(\frac{x}{y}\right)$.

2. Consider the ellipse given by the equation $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{81} = 1$.

   (a) Find the equation of the tangent line to the ellipse at the point (u, v) where $u = 4$ and $v > 0$.

   (b) Sketch the ellipse and the line to check your answer.

3. Find the derivative of $f(x) = \pi^{\tan^{-1}(\omega x)}$, where ω is a constant.

4. Let (a, b) be a point in the circle $x^2 + y^2 = 144$. Use implicit differentiation to find the slope of the tangent line to the circle at (a, b).

5. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

6. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

7. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

8. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

9. Use implicit differentiation to find the derivative of $\arctan(x)$. Thus if $x = \tan(y)$, use implicit differentiation to compute dy/dx. Can you simplify to express dy/dx in terms of x?

10. (a) Compute $\dfrac{d}{dx}\arcsin(\cos(x))$.

    (b) Compute $\frac{d}{dx}(\arcsin(x) + \arccos(x))$. Give a geometric explanation as to why the answer is 0.

    (c) Compute $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}(x)\right)$ and simplify to show that the derivative is 0. Give a geometric explanation of your result.

11. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

12. Let f be defined by $f(x) = e^{-x^2}$.

    (a) For which values of x is $f'(x) = 0$

    (b) For which values of x is $f''(x) = 0$

13. The notation $\tan^{-1}(x)$ is ambiguous. It is not clear if the exponent -1 indicates the reciprocal or the inverse function. If we allow both interpretations, how many different ways can you (correctly) compute the derivative $f'(x)$ for
$$f(x) = (\tan^{-1})^{-1}(x)?$$

    In order to avoid this ambiguity, we will generally use $\cot(x)$ for the reciprocal of $\tan(x)$ and $\arctan(x)$ for the inverse of the tangent function restricted to the domain $(-\pi/2, \pi/2)$.

Worksheet # 15: Related Rates

1. Let a and b denote the length in meters of the two legs of a right triangle. At time $t = 0$, $a = 20$ and $b = 20$. If a is decreasing at a constant rate of 2 meters per second and b is increasing at a constant rate of 3 meters per second. Find the rate of change of the area of the triangle at time $t = 5$ seconds.

2. A person 6 feet tall walks along a straight path at a rate of 4 feet per second away from a streetlight that is 15 feet above the ground. Assume that at time $t = 0$ the person is touching the streetlight.
   (a) Draw a picture to represent the situation.
   (b) Find an equation that relates the length of the person's shadow to the person's position (relative to the streetlight).
   (c) Find the rate of change in the length of the shadow when $t = 3$.
   (d) Find how fast is the tip of the person's shadow is moving when $t = 4$.
   (e) Does the precise time make a difference in these calculations?

3. A spherical snow ball is melting. The rate of change of the surface area of the snow ball is constant and equal to -7 square centimeters per hour. Find the rate of change of the radius of the snow ball when $r = 5$ centimeters.

4. The height of a cylinder is a linear function of its radius (i.e. $h = ar + b$ for some a, b constants). The height increases twice as fast as the radius r and $\frac{dr}{dt}$ is constant. At time $t = 1$ seconds the radius is $r = 1$ feet, the height is $h = 3$ feet and the rate of change of the volume is 16π cubic feet/second.
   (a) Find an equation to relate the height and radius of the cylinder.
   (b) Express the volume as a function of the radius.
   (c) Find the rate of change of the volume when the radius is 4 feet.

5. A water tank is shaped like a cone with the vertex pointing down. The height of the tank is 5 meters and diameter of the base is 2 meters. At time $t = 0$ the tank is full and starts to be emptied. After 3 minutes the height of the water is 4 meters and it is decreasing at a rate of 0.5 meters per minute. At this instant, find the rate of change of the volume of the water in the tank. What are the units for your answer? Recall that the volume of a right-circular cone whose base has radius r and of height h is given by $V = \frac{1}{3}\pi r^2 h$.

6. A plane flies at an altitude of 5000 meters and a speed of 360 kilometers per hour. The plane is flying in a straight line and passes directly over an observer.
   (a) Sketch a diagram that summarizes the information in the problem.
   (b) Find the angle of elevation 2 minutes after the plane passes over the observer.
   (c) Find rate of change of the angle of elevation 2 minutes after the plane passes over the observer.

7. A car moves at 50 miles per hour on a straight road. A house is 2 miles away from the road. What is the rate of change in the angle between the house and the car and the house and the road when the car passes the house.

8. A car moves along a road that is shaped like the parabola $y = x^2$. At what point on the parabola are the rates of change for the x and y coordinates equal?

9. Let $f(x) = \dfrac{1}{1 + x^3}$ and $h(x) = \dfrac{1}{1 + f(x)}$
   (a) Find $f'(x)$.
   (b) Use the previous result to find $h'(x)$.
   (c) Let $x = x(t)$ be a function of time t with $x(1) = 1$ and set $F(t) = h(x(t))$. If $F'(1) = 18$, find $x'(1)$.

Worksheet # 16: Review for Exam II

1. (a) State the definition of the derivative of a function $f(x)$ at a point a.

   (b) Find a function f and a number a such that

   $$\frac{f(x) - f(a)}{x - a} = \frac{\ln(2x-1)}{x-1}$$

   (c) Evaluate the following limit by using (a) and (b),

   $$\lim_{x \to 1} \frac{\ln(2x-1)}{x-1}$$

2. State the following rules with the hypotheses and conclusion.

   (a) The product rule and quotient rule.

   (b) The chain rule.

3. A particle is moving along a line so that at time t seconds, the particle is $s(t) = \frac{1}{3}t^3 - t^2 - 8t$ meters to the right of the origin.

   (a) Find the time interval(s) when the particle's velocity is negative.

   (b) Find the time(s) when the velocity is zero.

   (c) Find the time interval(s) when the particle's acceleration is positive.

   (d) Find the time interval(s) when the particle is speeding up. Hint: What do we need to know about velocity and acceleration in order to know that the derivative of the speed is positive?

4. Compute the first derivative of each of the following functions:

   (a) $f(x) = \cos(4\pi x^3) + \sin(3x+2)$

   (b) $b(x) = x^4 \cos(3x^2)$

   (c) $y(\theta) = e^{\sec(2\theta)}$

   (d) $k(x) = \ln(7x^2 + \sin(x) + 1)$

   (e) $u(x) = (\sin^{-1}(2x))^2$

   (f) $h(x) = \dfrac{8x^2 - 7x + 3}{\cos(2x)}$

   (g) $m(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}}$

   (h) $q(x) = \dfrac{e^x}{1+x^2}$

   (i) $n(x) = \cos(\tan(x))$

   (j) $w(x) = \arcsin(x) \cdot \arccos(x)$

5. Let $f(x) = \cos(2x)$. Find the fourth derivative at $x = 0$, $f^{(4)}(0)$.

6. Let f be a one to one, differentiable function such that $f(1) = 2$, $f(2) = 3$, $f'(1) = 4$ and $f'(2) = 5$. Find the derivative of the inverse function, $(f^{-1})'(2)$.

7. The tangent line to $f(x)$ at $x = 3$ is given by $y = 2x - 4$. Find the tangent line to $g(x) = \dfrac{x}{f(x)}$ at $x = 3$. Put your answer in slope-intercept form.

8. Consider the curve $xy^3 + 12x^2 + y^2 = 24$. Assume this equation can be used to define y as a function of x (i.e. $y = y(x)$) near $(1, 2)$ with $y(1) = 2$. Find the equation of the tangent line to this curve at $(1, 2)$.

9. Each side of a square is increasing at a rate of 5 cm/s. At what rate is the area of the square increasing when the area is 14 cm^2?

10. The sides of a rectangle are varying in such a way that the area is constant. At a certain instant the length of a rectangle is 16 m, the width is 12 m and the width is increasing at 3 m/s. What is the rate of change of the length at this instant?

11. Suppose f and g are differentiable functions such that $f(2) = 3$, $f'(2) = -1$, $g(2) = \frac{1}{4}$, and $g'(2) = 2$. Find:

    (a) $h'(2)$ where $h(x) = \ln([f(x)]^2)$;

    (b) $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

12. Abby is driving north along Ash Road. Boris driving west on Birch Road. At 11:57 am, Boris is 5 km east of Oakville and traveling west at a speed of 60 km/h and Abby is 10 km north of Oakville and traveling north at a speed of 50 km/h.

    (a) Make a sketch showing the location and direction of travel for Abby and Boris.

    (b) Find the rate of change of the distance between Abby and Boris at 11:57 AM.

    (c) At 11:57 AM, is the distance between Abby and Boris increasing, decreasing, or not changing?

Worksheet # 17: Linear Approximation and Applications

1. For each of the following, use a linear approximation to the change in the function and a convenient nearby point to estimate the value:
   (a) $(3.01)^3$
   (b) $\sqrt{17}$
   (c) $8.06^{2/3}$
   (d) $\tan(44°)$
2. What is the relation between the linearization of a function $f(x)$ at $x = a$ and the tangent line to the graph of the function $f(x)$ at $x = a$ on the graph?
3. Use the linearization of $\sqrt{x}$ at $x = 16$ to estimate $\sqrt{18}$;
   (a) Find a decimal approximation to $\sqrt{18}$ using a calculator.
   (b) Compute both the absolute error and the percentage error if we use the linearization to approximate $\sqrt{18}$.
4. Suppose we want to paint a sphere of radius 200 cm with a coat of paint 0.1 mm thick. Use a linear approximation to approximate the amount of paint we need to do the job.
5. Let $f(x) = \sqrt{16 + x}$. First, find the linearization to $f(x)$ at $x = 0$, then use the linearization to estimate $\sqrt{15.75}$. Present your solution as a rational number.
6. Find the linearization $L(x)$ to the function $f(x) = \sqrt{1 - 2x}$ at $x = -4$.
7. Find the linearization $L(x)$ to the function $f(x) = \sqrt[3]{x + 4}$ at $x = 4$, then use the linearization to estimate $\sqrt[3]{8.25}$.
8. Your physics professor tells you that you can replace $\sin(\theta)$ with θ when θ is close to zero. Explain why this is reasonable.
9. Suppose we measure the radius of a sphere as 10 cm with an accuracy of $\pm$ 0.2 cm. Use linear approximations to estimate the maximum error in:
   (a) the computed surface area.
   (b) the computed volume.
10. Suppose that $y = y(x)$ is a differentiable function which is defined near $x = 2$, satisfies $y(2) = -1$ and
$$x^2 + 3xy^2 + y^3 = 9.$$
    Use the linear approximation to the change in y to approximate the value of $y(1.91)$.
11. (Review) Can you find a function f so that $f'(x) = 2x + 1$? Can you find a function g so that $g'(x) = \sin(x) + \cos(x)$? Can you find more than one answer?
12. (Review) Find the derivative of $\sin(\sin(\sin(\sin(x))))$.

Worksheet # 18: Extreme Values and the Mean Value Theorem

1. Comprehension check:
   (a) True or False: If $f'(c) = 0$ then f has a local maximum or local minimum at c.
   (b) True or False: If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.
   (c) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.
   (d) True or False: If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.

2. (a) Define the following terms or concepts:
   - Critical point
   - f has a local maximum at $x = a$
   - Absolute maximum

   (b) State the following:
   - The First Derivative Test for Critical Points
   - The Mean Value Theorem

3. Sketch the following:
   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.
   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.
   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.
   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.
   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.

4. Find the critical points for the following functions:
   (a) $f(x) = x^4 + x^3 + 1$
   (b) $g(x) = e^{3x}(x^2 - 7)$
   (c) $h(x) = |5x - 1|$

5. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.
   (a) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$
   (b) $h(x) = x + \sqrt{1 - x^2}$, $[-1, 1]$

6. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.
      i. Find the critical point(s) of $f(x)$.
      ii. Find the intervals on which $f(x)$ is increasing or decreasing.
      iii. Find the local extrema of $f(x)$.

   (b) Repeat with the function $f(x) = \dfrac{x}{x^2 + 4}$ on $(-\infty, \infty)$.
   (c) Repeat with the function $f(x) = \sin^2(x) - \cos(x)$ on $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$.

7. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.
   (a) $f(x) = \dfrac{x}{x + 2}$ on the interval $[1, 4]$
   (b) $f(x) = \sin(x) - \cos(x)$ on the interval $[0, 2\pi]$

8. Use the mean value theorem to show that $\sin(x) \leq x$ for $x \geq 0$. What can you say for $x \leq 0$?

Worksheet # 19: The Shape of a Graph

1. Explain how to use the second derivative test to identify and classify local extrema of a twice differentiable function $f(x)$. Does the test always work? What should you do if it fails?

2. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.

3. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why did she deserve the ticket?

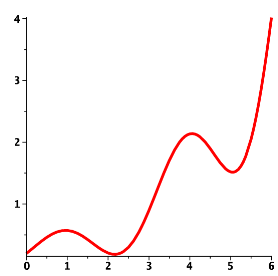
4. (a) Consider the function $f(x) = x^4 - 8x^3 + 5$.
   - i. Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
   - ii. Find the intervals of concavity of $f(x)$.
   - iii. Find the points of inflection of $f(x)$.

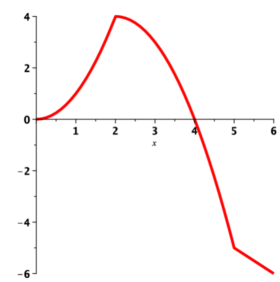
   (b) Repeat with the function $f(x) = 2x + \sin(x)$ on $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

   (c) Repeat with the function $f(x) = x + \frac{4}{x}$.

   (d) Repeat with the function $f(x) = xe^x$.

5. Below are the graphs of two functions.





   (a) Find the intervals where each function is increasing and decreasing respectively.

   (b) Find the intervals of concavity for each function.

   (c) For each function, identify all local extrema and inflection points on the interval (0,6).

6. Find the local extrema of the following functions using the second derivative test:

   (a) $f(x) = x^5 - 5x + 4$

   (b) $g(x) = 5x - 10\ln(2x)$

7. Find the local extrema of $f(x) = 3x^5 - 5x^3 + 10$ using the second derivative test where possible.

8. Sketch a graph of a continuous function $f(x)$ with the following properties:

   - f is increasing on $(-\infty, -3) \cup (1, 7) \cup (7, \infty)$
   - f is decreasing on $(-3, 1)$
   - f is concave up on $(0, 3) \cup (7, \infty)$
   - f is concave down on $(-\infty, 0) \cup (3, 7)$

Worksheet # 20: Limits at Infinity & L'Hôpital's Rule

1. (a) Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.

   (b) Explain the difference between "$\lim_{x\to-3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".

2. Evaluate the following limits, or explain why the limit does not exist:

   (a) $\lim_{x\to\infty} \dfrac{3x^2 - 7x}{x - 8}$

   (b) $\lim_{x\to\infty} \dfrac{2x^2 - 6}{x^4 - 8x + 9}$

   (c) $\lim_{x\to-\infty} \dfrac{x}{x^6 - 4x^2}$

   (d) $\lim_{x\to-\infty} 3$

   (e) $\lim_{x\to\pm\infty} \dfrac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$

   (f) $\lim_{x\to-\infty} \dfrac{\sqrt{x^{10} + 2x}}{x^5}$

3. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \left(\dfrac{x^2}{x+1} - \dfrac{x^2}{x-1}\right)$.

4. Sketch a graph with all of the following properties:

   - $\lim_{t\to\infty} f(t) = 2$
   - $\lim_{t\to-\infty} f(t) = 0$
   - $\lim_{t\to0^+} f(t) = \infty$
   - $\lim_{t\to0^-} f(t) = -\infty$
   - $\lim_{t\to4} f(t) = 3$
   - $f(4) = 6$

5. Find the following limits;

   (a) $\lim_{x\to\infty} \dfrac{3x + 2\sqrt{x}}{1 - x}$

   (b) $\lim_{x\to-\infty} \dfrac{2x - 5}{|3x + 2|}$

   (c) $\lim_{x\to\infty} \dfrac{5x^2 + \sin x}{3x^2 + \cos x}$

6. Carefully state l'Hôpital's Rule.

7. Compute the following limits. Use l'Hôpital's Rule where appropriate but first check that no easier method will solve the problem.

   (a) $\lim_{x\to1} \dfrac{x^9 - 1}{x^5 - 1}$

   (b) $\lim_{x\to0} \dfrac{\sin(4x)}{\tan(5x)}$

   (c) $\lim_{x\to2} \dfrac{x^2 + x - 6}{x - 2}$

   (d) $\lim_{x\to1} \dfrac{x^2 + 2x - 2}{x^2 - 2x + 2}$

8. Find the value A for which we can use l'Hôpital's rule to evaluate the limit

$$\lim_{x\to2} \frac{x^2 + Ax - 2}{x - 2}.$$

   For this value of A, give the value of the limit.

9. Compute the following limits. Use l'Hôpital's Rule where appropriate but first check that no easier method will solve the problem.

(a) $\lim_{x \to -\infty} x^2 e^x$

(b) $\lim_{x \to \infty} x^3 e^{-x^2}$

(c) $\lim_{x \to \pi} \frac{\cos(x) + 1}{x^2 - \pi^2}$

(d) $\lim_{x \to \infty} x \cdot \left(\arctan(x) - \frac{\pi}{2} \right)$

Worksheet # 21: Optimization

1. Suppose that f is a function on an open interval $I = (a, b)$ and c is in I. Suppose that f is continuous at c, $f'(x) > 0$ for $x > c$ and $f'(x) < 0$ for $x < c$. Is $f(c)$ an absolute minimum value for f on I? Justify your answer.

2. Find the dimensions of x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating x and y?

   (b) Find a formula for the area in terms of x alone.

   (c) Solve the optimization problem.

3. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

4. A hockey team plays in an arena with a seating capacity of 15000 spectators. With the ticket price set at \$12, average attendance at a game has been 11000. A market survey indicates that for each dollar the ticket price is lowered, average attendance will increase by 1000. How should the owners of the team set the ticket price to maximize their revenue from ticket sales?

5. An oil company needs to run a pipeline to a nearby station. The station and oil company are on opposite sides of a river that is 1 km wide, and that runs exactly west-east and the station is 10 km east along the river from the the oil company. The cost of building pipe on land is \$200 per meter and the cost of building pipe in water is \$300 per meter. Set up an equation whose solution(s) are the critical points of the cost function for this problem.

   Find the least expensive way to construct the pipe.

6. A flexible tube of length 4 m is bent into an L-shape. Where should the bend be made to minimize the distance between the two ends?

7. A 10 meter length of rope is to be cut into two pieces to form a square and a circle. How should the rope be cut to maximize the enclosed area?

8. Find the point(s) on the hyperbola $y = \dfrac{16}{x}$ that is (are) closest to $(0, 0)$. Be sure to clearly state what function you choose to minimize or maximize and why.

9. Consider a can in the shape of a right circular cylinder. The top and bottom of the can is made of a material that costs 4 cents per square centimeter, and the side is made of a material that costs 3 cents per square centimeter. We want to find the dimensions of the can which has volume 72 π cubic centimeters, and whose cost is as small as possible.

   (a) Find a function $f(r)$ which gives the cost of the can in terms of radius r. Be sure to specify the domain.

   (b) Give the radius and height of the can with least cost.

   (c) Explain how you known you have found the can of least cost.

10. A box is to have a square base, no top, and a volume of 10 cubic centimeters. What are the dimensions of the box with the smallest possible total surface area? Provide an exact answer; do not convert your answer to decimal form. Make a sketch and introduce all the notation you are using.

Worksheet # 22: Newton's Method and Antiderivatives

1. Use Newton's method to find an approximation to $\sqrt[3]{2}$. You may do this by finding a solution of $x^3 - 2 = 0$.
2. Use Newton's method to approximate the critical points of the function $f(x) = x^5 - 7x^2 + x$.
3. Let $f(x) = \dfrac{x}{1+x^2}$.
   (a) Solve $f(x) = 0$ without using Newton's method.
   (b) Use Newton's method to solve $f(x) = 0$ beginning with the starting point $x_0 = 2$. Does something interesting happen?
   (c) Make a sketch of the graph of f and explain what you observed in part b).
4. (a) Let $f(x) = \dfrac{x^3}{3} + 1$. Calculate the derivative $f'(x)$. Give an anti-derivative of $f'(x)$. Give an anti-derivative that is different from $f(x)$.
   (b) Let $g(x) = x^2 + 1$. Let $G(x)$ be any anti-derivative of g. What is $G'(x)$?
5. Find f given that
$$f'(x) = \sin(x) - \sec(x)\tan(x), \qquad f(\pi) = 1.$$
6. Find g given that
$$g''(t) = -9.8, \qquad g'(0) = 1, \qquad g(0) = 2.$$
   On the surface of the earth, the acceleration of an object due to gravity is approximately $-9.8\,\mathrm{m/s^2}$. What situation could we describe using the function g? Be sure to specify what g and its first two derivatives represent.
7. A small rock is dropped from a bridge and the splash is heard 3 seconds later. How high is the bridge?
8. Let f be a function on the domain $(-\infty, \infty)$ that satisfies $(f')^2 = 1$. This is an example of a *differential equation*. Suppose also that we are given an *initial value condition* $f(0) = 1$.
   (a) Show that this does not have a unique solution by finding two different functions that satisfy both conditions.
   (b) What does the fact that there are multiple solutions say about this as a model for physical phenomena?
9. Find a function $f(x)$ such that $f'(x) = f(x)$. Find the solution, given initial condition $f(0) = \pi$.
10. Let $f(x) = 1/x$, $F(x) = \ln(|x|)$, and
$$G(x) = \begin{cases} \ln(x), & x > 0 \\ \ln(-x) + 8, & x < 0. \end{cases}$$
   (a) Is F an anti-derivative of f? Is G an anti-derivative of f? Is $F - G$ equal to a constant?
   (b) Does Theorem 1 on page 275 imply that $F - G$ is constant? Is the theorem wrong?

Worksheet # 23: Approximating Area

1. Write each of following in summation notation:

   (a) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

   (b) $2 + 4 + 6 + 8 + 10 + 12 + 14$

   (c) $2 + 4 + 8 + 16 + 32 + 64 + 128$.

2. Compute $\displaystyle\sum_{i=1}^{4}\left(\sum_{j=1}^{3}(i+j)\right)$.

   The following summation formulas will be useful below.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Find the number n such that $\sum_{i=1}^{n} i = 78$.

4. Give the value of the following sums.

   (a) $\displaystyle\sum_{k=1}^{20}(2k^2+3)$

   (b) $\displaystyle\sum_{k=11}^{20}(3k+2)$

5. The velocity of a train at several times is shown in the table below. Assume that the velocity changes linearly between each time given.

t=time in minutes	0	3	6	9
v(t)=velocity in Km/h	20	80	100	140

   (a) Plot the velocity of the train versus time.

   (b) Compute the left and right-endpoint approximations to the area under the graph of v.

   (c) Explain why these approximate areas are also an approximation to the distance that the train travels.

6. Let $f(x) = 1/x$. Divide the interval $[1, 3]$ into five subintervals of equal length and compute R_5 and L_5, the left and right endpoint approximations to the area under the graph of f in the interval $[1, 3]$. Is R_5 larger or smaller than the true area? Is L_5 larger or smaller than the true area?

7. Let $f(x) = \sqrt{1 - x^2}$. Divide the interval $[0, 1]$ into four equal subintervals and compute L_4 and R_4, the left and right-endpoint approximations to the area under the graph of f. Is R_4 larger or smaller than the true area? Is L_4 larger or smaller than the true area? What can you conclude about the value π?

8. Let $f(x) = x^2$.

   (a) If we divide the interval $[0, 2]$ into n equal intervals of equal length, how long is each interval?

   (b) Write a sum which gives the right-endpoint approximation R_n to the the area under the graph of f on $[0, 2]$.

   (c) Use one of the formulae for the sums of powers of k to find a closed form expression for R_n.

   (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.

Worksheet # 24: Review for Exam III

1. (a) Find the linearization, $L(x)$, to $f(x) = \sqrt{1-2x}$ at $x = -4$.
   (b) Use the result of (a) to approximate $\sqrt{11}$.
   (c) Find the absolute error in the approximation of $\sqrt{11}$ by using your calculator.

2. (a) Describe in words and diagrams how to use the first and second derivative tests to identify and classify extrema of a function $f(x)$.
   (b) Use the first derivative test to identify and classify the extrema of the function

$$f(x) = 2x^3 + 3x^2 - 72x - 47.$$

3. Find the absolute minimum of the function $f(t) = t + \sqrt{1-t^2}$ on the interval $[-1, 1]$. Be sure to specify the value of t where the minimum is attained.

4. For each of the following functions (i) Find the intervals on which f is increasing or decreasing. (ii) Find the local maximum and minimum values of f. (iii) Find the intervals of concavity and the inflection points.

   (a) $f(x) = x^4 - 2x^2 + 3$
   (b) $f(x) = e^{2x} + e^{-x}$

5. For what values of c does the polynomial $p(x) = x^4 + cx^3 + x^2$ have two inflection points? One inflection point? No inflection points?

6. (a) State the Mean Value Theorem. Use complete sentences.
   (b) Does there exist a function f such that $f(0) = -1, f(2) = 4$, and $f'(x) \leq 2$ for all x?

7. Evaluate the following limits;

   (a) $\displaystyle\lim_{x\to\infty} \frac{7x^8 + 3x^3 - 1}{21x^3 - 13x^8 + x^2}$
   (b) $\displaystyle\lim_{x\to\infty} \frac{2012 - x^{2012}}{2013 - x^{2013}}$
   (c) $\displaystyle\lim_{x\to-\infty} \frac{2012 - x^{2012}}{2013 - x^{2013}}$
   (d) $\displaystyle\lim_{x\to-\infty} \frac{5x}{\sqrt{9x^2+1}}$

8. (a) State L'Hopital's Rule for limits in indeterminate form of type $0/0$. Use complete sentences, and include all necessary assumptions.
   (b) Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$
   (c) Evaluate $\displaystyle\lim_{x\to 0^+} x^3 \ln(x)$
   (d) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{x+2}{\sqrt{9x^2+1}}$
   (e) Evaluate $\displaystyle\lim_{x\to 2} \frac{e^{2x}}{x+2}$

9. A poster is to have an area of 180 cm^2 with 1 cm margins at the bottom and sides and 2 cm margins at the top. What dimensions will give the largest printed area. Be sure to explain how you know you have found the largest area.

   (a) Draw a picture and write the constraint equation.
   (b) Write the function you are asked to maximize or minimize and determine its domain.
   (c) Find the maximum or minimum of the function that you found in part (b).

10. Find a positive number such that the sum of the number and twice its reciprocal is small as possible.

11. Let $f(x) = x^2 - 3x + 1$, $x_1 = 3$. Apply Newton's Method to $f(x)$ and initial guess x_1 to calculate x_2, x_3, x_4.

12. Find the most general anti-derivative of $f(x) = x^2 + \cos(2x + 1)$.

13. Find a function with $f''(x) = \sin(2x)$, $f(\pi) = 1$, and $f'(0) = 2$.

14. Consider the region bounded by the graph of $f(x) = 1/x$, the x-axis, and the lines $x = 1$ and $x = 2$. Find L_3, the left endpoint approximation of this area with 3 subdivisions.

15. We know $\sum_{k=1}^{n} a_k = n^2 + 2n$ for $n = 1, 2, \ldots$. Find $\sum_{k=1}^{20}(4a_k + 1)$. Find $\sum_{k=5}^{10} a_k$.

Worksheet # 25: Definite Integrals

1. Suppose $\int_0^1 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = 3$, $\int_0^1 g(x)\,dx = -1$, and $\int_0^2 g(x)\,dx = 4$.
   Compute the following using the properties of definite integrals:

   (a) $\displaystyle\int_1^2 g(x)\,dx$

   (b) $\displaystyle\int_0^2 [2f(x) - 3g(x)]\,dx$

   (c) $\displaystyle\int_1^1 g(x)\,dx$

   (d) $\displaystyle\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx$

   (e) $\displaystyle\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx$

2. Suppose that f is a continuous function and $6 \le f(x) \le 7$ for x in the interval $[3, 9]$.

   (a) Find the largest and smallest possible values for $\int_3^9 f(x)\,dx$.

   (b) Find the largest and smallest possible values for $\int_8^4 f(x)\,dx$.

3. Suppose that we know $\int_0^x f(t)\,dt = \sin(x)$, find the following integrals.

   (a) $\displaystyle\int_0^\pi f(t)\,dt$

   (b) $\displaystyle\int_{\pi/2}^\pi f(t)\,dt$

   (c) $\displaystyle\int_{-\pi}^\pi f(t)\,dt$

4. Find $\displaystyle\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \ge 3 \end{cases}$.

5. Recognize the sum as a Riemann sum and express the limit as an integral.

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{i^3}{n^4}$$

6. Let $f(x) = x$ and consider the partition $P = \{x_0, x_1, \ldots, x_n\}$ which divides the interval $[1, 3]$ into n subintervals of equal length.

   (a) Find a formula for x_k in terms of k and n.

   (b) We form a rectangle whose width is $(x_k - x_{k-1})$ and whose height is $f(x_k)$. Give the area of the rectangle.

   (c) Choose the sample points to be the right endpoint of each subinterval and thus $C = \{x_1, x_2, \ldots, x_n\}$. Form the Riemann sum $R(f, P, C)$ and use the formulae for sums of powers to simplify the Riemann sum.

   (d) Take the limit as n tends to infinity to find the area of the region $S = \{(x, y) : 1 \le x \le 3, 0 \le y \le x\}$.

   (e) Find the area of S using geometry to check your answer.

7. Simplify

$$\int_a^b f(t)\,dt + \int_b^c f(u)\,du + \int_c^a f(v)\,dv.$$

Worksheet # 26: The Fundamental Theorems of Calculus and the Net Change Theorem

1. (a) State both parts of the Fundamental Theorem of Calculus using complete sentences.

   (b) Consider the function $f(x)$ on $[1, \infty)$ defined by $f(x) = \int_1^x \sqrt{t^5 - 1}\, dt$. Find the derivative of f. Explain why the function f is increasing.

   (c) Find the derivative of the function $g(x) = \int_1^{x^3} \sqrt{t^5 - 1}\, dt$ on $(1, \infty)$.

2. Use Part I of the Fundamental Theorem of Calculus to evaluate the following integrals or explain why the theorem does not apply:

   (a) $\displaystyle\int_{-2}^{5} 6x\, dx$

   (b) $\displaystyle\int_{-2}^{7} \frac{1}{x^5}\, dx$

   (c) $\displaystyle\int_{-1}^{1} e^{u+1}\, du$

   (d) $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec^2(t)\, dt$

   (e) $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\sin(2x)}{\sin(x)}\, dx$

3. Use Part II of the Fundamental Theorem of Calculus to find the derivative of the following functions:

   (a) $g(x) = \displaystyle\int_1^x \left(2 + t^4\right)^5\, dt$

   (b) $F(x) = \displaystyle\int_x^4 \cos\left(t^5\right)\, dt$

   (c) $h(x) = \displaystyle\int_0^{x^2} \sqrt[3]{1 + r^3}\, dr$

   (d) $y(x) = \displaystyle\int_{\frac{1}{x^2}}^{0} \sin^3(t)\, dt$

   (e) $G(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \sqrt{t}\sin(t)\, dt$

4. A population of rabbits at time t increases at a rate of $40 - 12t + 3t^2$ rabbits per year where t is measured in years. Find the population after 8 years if there are 10 rabbits at $t = 0$.

5. Suppose the velocity of a particle traveling along the x-axis is given by $v(t) = 3t^2 + 8t + 15$ m/s at time t seconds. The particle is initially located 5 meters left of the origin. How far does the particle travel from $t = 2$ seconds to $t = 3$ seconds? After 3 seconds, where is the particle with respect to the origin?

6. Suppose an object traveling in a straight line has a velocity function given by $v(t) = t^2 - 8t + 15$ km/hr. Find the displacement and distance traveled by the object from $t = 2$ to $t = 4$ hours.

7. An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?

8. Recognize each of the sums as a Riemann sum, express the limit as an integral and use the Fundamental Theorem to evaluate the limit.

   (a) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\sqrt{3 + \frac{i}{n}}}{n}$

   (b) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} 2\frac{(2 + \frac{2i}{n})^2}{n}$

Worksheet # 27: Substitution and Further Transcendental Functions

1. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int \frac{4}{(1+2x)^3}\,dx$

   (b) $\displaystyle\int x^2\sqrt{x^3+1}\,dx$

   (c) $\displaystyle\int \cos^4(\theta)\sin(\theta)\,d\theta$

   (d) $\displaystyle\int \sec^3(x)\tan(x)\,dx$

   (e) $\displaystyle\int e^x \sin\left(e^x\right)\,dx$

   (f) $\displaystyle\int \frac{2x+3}{x^2+3x}\,dx$

2. Evaluate the following definite integrals, and indicate any substitutions that you use:

   (a) $\displaystyle\int_0^7 \sqrt{4+3x}\,dx$

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cos\left(\sin(x)\right)\,dx$

   (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x^2}}\,dx$

   (d) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

   (e) $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

3. Assume f is a continuous function.

   (a) If $\displaystyle\int_0^9 f(x)\,dx = 4$, find $\displaystyle\int_0^3 x\cdot f(x^2)\,dx$.

   (b) If $\displaystyle\int_0^u f(x)\,dx = 1+e^{u^2}$ for all real numbers u, find $\displaystyle\int_0^2 f(2x)\,dx$.

4. Evaluate the following indefinite integrals:

   (a) $\displaystyle\int \frac{dx}{x}$

   (b) $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

   (c) $\displaystyle\int \frac{dt}{1+t^2}$

   (d) $\displaystyle\int \frac{dv}{|v|\sqrt{v^2-1}}$

   (e) $\displaystyle\int e^x\,dx$

   (f) $\displaystyle\int 2e^{2x}\,dx$

5. Use the equation $b^x = e^{x\ln(b)}$ to find the indefinite integral $\displaystyle\int b^x\,dx$

6. Find b so that $\displaystyle\int_1^b \frac{dx}{x}$ is equal to

   (a) $\ln(3)$

   (b) 3

7. Find b such that $\displaystyle\int_0^b \frac{dx}{1+x^2} = \frac{\pi}{3}$.

8. Which integral should be evaluated using substitution? Evaluate both integrals:

(a) $\displaystyle\int \frac{9\,dx}{1+x^2}$

(b) $\displaystyle\int \frac{x\,dx}{1+9x^2}$

9. Find a so that if $x = au$, then $\sqrt{16+x^2} = 4\sqrt{1+u^2}$.

10. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

(a) $\displaystyle\int \frac{dx}{x^2+3}$

(b) $\displaystyle\int \frac{\cos(\ln(t))\,dt}{t}$

(c) $\displaystyle\int \frac{x\,dx}{\sqrt{7-x^2}}$

(d) $\displaystyle\int \frac{dt}{4t^2+9}$

(e) $\displaystyle\int \frac{\ln(\arccos(x))\,dx}{\arccos(x)\sqrt{1-x^2}}$

(f) $\displaystyle\int \frac{dt}{|t|\sqrt{12t^2-3}}$

(g) $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$

(h) $\displaystyle\int e^{9-2x}\,dx$

11. Evaluate the following definite integrals, and indicate any substitutions that you use:

(a) $\displaystyle\int_{\tan(.5)}^{\tan(1.5)} \frac{dx}{x^2+1}$

(b) $\displaystyle\int_{-e^2}^{-e} \frac{dt}{t}$

(c) $\displaystyle\int_{-1/5}^{1/5} \frac{dx}{\sqrt{4-25x^2}}$

(d) $\displaystyle\int_{1}^{\sqrt{3}} \frac{dx}{\arctan(x)(1+x^2)}$

(e) $\displaystyle\int_{0}^{4} \frac{dt}{4t^2+9}$

(f) $\displaystyle\int_{1/(2\sqrt{2})}^{1/2} \frac{dx}{|x|\sqrt{16x^2-1}}$

Worksheet #28: Exponential Growth and Decay, Area Between Curves

1. Solve the following equations for α:

   (a) $500 = 1000e^{20\alpha}$

   (b) $40 = \alpha e^{10k}$, where $k = \frac{\ln(2)}{7}$.

   (c) $100,000 = 40,000e^{0.06\alpha}$.

   (d) $\alpha = 2,000e^{36k}$, where $k = \frac{\ln(0.5)}{18}$.

2. The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

3. A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially and 800 after 24 hours, how many cells will there be after a further 12 hours?

4. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.

   $$\frac{dP}{dt} = KP$$

   where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$

   (a) Give the units of K.

   (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$. How many mosquitoes will there be after 90 days?

5. Suppose that $P(t)$ gives the number of individuals in a population at time t where t is measured in years. Each year 23 out of 1000 individuals give birth and 11 out of 1000 individuals die.

   Find a differential equation of the form $P' = kP$ that the function P satisfies.

6. Suppose that f is a solution of the differential equation $f' = kf$ on an open interval (a, b) where k is a constant. Compute the derivative of $g(x) = e^{-kx}f(x)$ and show that g is constant.

   Explain why $f(x) = Ae^{kx}$?

7. Find the area of the region between the graphs of $y = x^2$ and $y = x^3$.

8. Find the area of the regions enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x + \frac{3}{4}$ in two ways. 1) By writing an integral in x. 2) Solve each equation to express x in terms of y and write an integral with respect to y.

9. Find the area of the region enclosed by the graphs of $y = x + 1$ and $y = x^3 + x^2 - x + 1$.

10. (a) Show that $\lim_{h\to 0} \ln\left((1 + hr)^{t/h}\right) = rt$. (Hint: Use L'Hospital).

    (b) From (a) show that $\lim_{h\to 0} \left((1 + hr)^{t/h}\right) = e^{rt}$. (Hint: Use your previous result and properties of logarithms.)

    (c) The compound interest formula is

    $$P(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

where $P(t)$ is the value at time t, A_0 is the initial investment, r is the interest rate, t is the time in years, and n is the number of times compounded per year. Verify that

$$\lim_{h \to 0} \left((1 + hr)^{t/h} \right) = \lim_{n \to \infty} \left(1 + \frac{r}{n} \right)^{nt}$$

(Hint: Use the change of variable $h = \frac{1}{n}$)

Use part (b) to derive a formula involving e for continuously compounded interest.

(Hint: How can you represent continuously compounded interest as a limit where the interval of time between each compounding decreases to zero and the number of compoundings increases to infinity?)

Worksheet # 29: Area Between Curves, Review I for Final

1. Find the area of the region between the graphs of $y = x^2$ and $y = x^3$.

2. Find the area of the regions enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x + \frac{3}{4}$ in two ways. 1) By writing an integral in x. 2) Solve each equation to express x in terms of y and write an integral with respect to y.

3. Find the area of the region enclosed by the graphs of $y = x + 1$ and $y = x^3 + x^2 - x + 1$.

4. Compute the derivative of the given function:

   (a) $f(\theta) = \cos(2\theta^2 + \theta + 2)$

   (b) $g(u) = \ln\left(\sin^2(u)\right)$

   (c) $h(x) = \displaystyle\int_{-3599}^{x} (t^2 - te^{t^2+t+1})\, dt$

   (d) $r(y) = \arccos(y^3 + 1)$

5. Compute the following definite integrals:

   (a) $\displaystyle\int_0^{\pi} \sec^2(t/4)\, dt$

   (b) $\displaystyle\int_0^{1} xe^{-x^2}\, dx$

   (c) $\displaystyle\int_0^{1} x(1+x)^6\, dx$

   (d) $\displaystyle\int_0^{\pi/4} \sin(2x)\cos(2x)\, dx$

6. What is the area of the bounded region bounded by $f(x) = \frac{1}{x}$, $x = e^2$, $x = e^8$ and x-axis? Sketching the region might be helpful.

7. If $F(x) = \displaystyle\int_{3x^2+1}^{7} \cos(t^2)\, dt$, find $F'(x)$. Justify your work.

8. Suppose a bacteria colony grows at a rate of $r(t) = 100\, e^{0.02t}$ with t given in hours. What is the growth in population from time $t = 1$ to $t = 3$?

9. Use the left endpoint approximation with 4 equal subintervals to estimate the value of $\displaystyle\int_1^5 x^2\, dx$. Will this estimate be larger or smaller than the actual value of definite integral? Explain your answer.

10. Find an antiderivative for the function $f(x) = \dfrac{x - 4x^3}{1 + x^4}$.

11. Give the interval(s) for which the function F is increasing. The function F is defined by

$$F(x) = \int_0^x \frac{5t - 3}{t^2 + 10}\, dt$$

12. Find a function $f(x)$ such that $f(e) = 0$ and $f'(x) = \dfrac{e^x - e^{-x}}{\ln x}$. (Hint: Consider the Fundamental Theorem of Calculus)

13. Which of the following is an antiderivative for the function $f(x) = e^x \sin(e^x)$. Circle all the correct answers.

    (a) $F(x) = -\cos(e^x)$

    (b) $F(x) = \sin(e^x)$

    (c) $F(x) = \int_0^x e^t \cos(e^t)\, dt$

    (d) $F(x) = \int_0^{e^x} \cos(t)\, dt$

    (e) $F(x) = \int_0^x e^t \sin(e^t)\, dt$

    (f) $F(x) = \int_0^{e^x} \sin(t)\, dt$

14. Which of the following integrals are the same as $\int_{1/2}^{1} \frac{\ln(\arcsin(x))}{\arcsin(x)\sqrt{1-x^2}}\,dx$. Circle all the correct answers.
(Hint: Use substitution method. You may need to do substitution more than once.)

(a) $\int_{1/2}^{1} \frac{\ln(u)}{u}\,du$

(b) $\int_{\pi/6}^{\pi/2} \frac{\ln(u)}{u}\,du$

(c) $\int_{1/2}^{1} t\,dt$

(d) $\int_{\ln(\pi/6)}^{\ln(\pi/2)} t\,dt$

(e) $\int_{\pi/6}^{\pi/2} t\,dt$

15. Compute the indefinite integral $\int \frac{\sin(x)}{1+\cos^2(x)}\,dx$.

16. Let $F(x) = \int_0^x \sin^2(t)\,dt$. Evaluate the limit

$$\lim_{x\to 0} \frac{F(x)}{x^2}.$$

17. Evaluate $\frac{d}{dx}(x^5 \int_2^{x^5} \frac{\sin(t)}{t}\,dt)$.

Worksheet # 30: Review II for Final

1. Compute the following limits.

   (a) $\lim_{t\to 9} \dfrac{9-t}{3-\sqrt{t}}$

   (b) $\lim_{\theta\to 0} \dfrac{\sin(3\theta)}{10\theta}$

   (c) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$

   (d) $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$

2. (a) State the limit definition of the continuity of a function f at $x = a$.

   (b) State the limit definition of the derivative of a function f at $x = a$.

   (c) Given $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4-3x & \text{if } x \geq 1 \end{cases}$. Is the function continuous at $x = 1$? Is the function differentiable at $x = 1$? Use the definition of the derivative. Graph the function to check your answer.

3. Provide the most general antiderivative of the following functions:

   (a) $f(x) = x^4 + x^2 + x + 1000$

   (b) $g(x) = (3x-2)^{20}$

   (c) $h(x) = \dfrac{\sin(\ln(x))}{x}$

4. Use implicit differentiation to find $\dfrac{dy}{dx}$, and compute the slope of the tangent line at (1,2) for the following curves:

   (a) $x^2 + xy + y^2 + 9x = 16$

   (b) $x^2 + 2xy - y^2 + x = 2$

5. An rock is thrown up the in the air and returns to the ground 4 seconds later. What is the initial velocity? What is the maximum height of the rock? Assume that the rock's motion is determined by the acceleration of gravity, 9.8 meters/second^2.

6. A conical tank with radius 5 meters and height 10 meters is being filled with water at a rate of 3 cubic meters per minute. How fast is the water level increasing when the water's depth is 3 meters?

7. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of the container is twice its width. Material for the base costs \$10 per square meter while material for the sides costs \$6 per square meter. Find the cost of materials for the least expensive possible container.

8. (a) State the Mean Value Theorem.

   (b) If $3 \leq f'(x) \leq 5$ for all x, find the maximum possible value for $f(8) - f(2)$.

9. Use linearization to approximate $\cos(\frac{11\pi}{60})$

   (a) Write down $L(x)$ at an appropriate point $x = a$ for a suitable function $f(x)$.

   (b) Use part(a) to find an approximation for $\cos(\frac{11\pi}{60})$

   (c) Find the absolute error in your approximation.

10. Find the value(s) c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} (cx)^3 & \text{if } x < 2 \\ \ln(x^c) & \text{if } x \geq 2 \end{cases}$$

11. (a) Find y' if $x^3 + y^3 = 6xy$.
    (b) Find the equation of the tangent line at $(3, 3)$.

12. Show that the function $f(x) = 3x^5 - 20x^3 + 60x$ has no absolute maximum or minimum.

13. Compute the following definite integrals:

    (a) $\int_{-1}^{1} e^{u+1}\, du$

    (b) $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$

    (c) $\int_{1}^{9} \frac{x-1}{\sqrt{x}}\, dx$

    (d) $\int_{0}^{10} |x - 5|\, dx$

    Hint: For some of the integrals, you will need to interpret the integral as an area and use facts from geometry to compute the integral.

14. Write as a single integral in the form $\int_a^b f(x)\, dx$:

$$\int_{-2}^{2} f(x)\, dx + \int_{2}^{5} f(x)\, dx - \int_{-2}^{-1} f(x)\, dx$$