MA 114 Worksheet 17: Average Value of a Function

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$
   (b) $f(x) = x^3,\ [-1, 1]$
   (c) $f(x) = \cos(x),\ \left[0, \frac{\pi}{6}\right]$
   (d) $f(x) = \frac{1}{x^2+1},\ [-1, 1]$
   (e) $f(x) = \frac{\sin(\pi/x)}{x^2},\ [1, 2]$
   (f) $f(x) = e^{-nx},\ [-1, 1]$
   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$
   (h) $f(x) = x^n$ for $n \geq 0$, $[0, 1]$

3. In a certain city the temperature (in $^\circ F$) t hours after 9 am was modeled by the function $T(t) = 50 + 14\sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9 am to 9 pm.

4. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

   where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity (with respect to r) over the interval $0 < r < R$. Compare the average velocity with the maximum velocity.

5. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t. Then use this new function to compute the average volume of inhaled air in the lungs in one respiratory cycle.

MA 114 Worksheet 18: Volumes I

1. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

2. Calculate the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$, where cross sections perpendicular to the y-axis are squares.

3. The base of a solid is the triangle with vertices at $(10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

4. Calculate the volume of the solid whose base is a circle of radius r centered at the origin and which has square cross sections perpendicular to the x-axis.

5. Calculate the volume of the following solid. The base is the parabolic region bounded by $y = x^2$ and $y = 4$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.

6. Sketch the solid whose volume is given by the integral
$$\pi \int_0^1 (y^2 + 1)^2 - 1\, dy.$$

7. Use disks or washers to find the volume of the solid obtained by rotating the given region about the specified line.
   (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.
   (b) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.
   (c) Compare the volumes you found in parts (a) and (b). Which is bigger? Why?

8. For each of the following, set up **but do not evaluate** an integral expression for the volume of the solid obtained by rotating the given region about the specified line.
   (a) R is the region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$; about the x-axis.
   (b) R is the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the y-axis.
   (c) R is the region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.
   (d) R is the region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

9. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

MA 114 Worksheet 19: Volumes II

1. (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y-axis using the method of cylindrical shells.

   (b) If you use the disk method to compute the same volume, are you integrating with respect to x or y? Why?

2. Sketch the enclosed region and use the shell method to calculate the volume of the solid obtained by rotating the region about the y-axis.

   (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$

   (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$

   (c) $y = 8 - x^3$, $y = 8 - 4x$, for $x \geq 0$

3. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the y-axis. Soda is extracted from the glass through a straw at the rate of $1/2$ cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

MA 114 Worksheet 20: Arc Length and Surface Area

1. (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.

   (b) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the x-axis. Include the required conditions on $f(x)$.

2. Find an integral expression for the arc length of the following curves. Do ***not*** evaluate the integrals.

   (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$.

   (b) $f(x) = x^4$ from $x = 2$ to $x = 6$.

   (c) $x^2 + y^2 = 1$

3. Find the arc length of the following curves.

   (a) $f(x) = x^{3/2}$ from $x = 100$ to $x = 101$.

   (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$.

   (c) $f(x) = e^x$ from $x = 0$ to $x = 1$.

4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.

5. Compute the surface areas of revolution about the x-axis over the given interval for the following functions.

   (a) $y = x$, $[0, 4]$

   (b) $y = x^3$, $[0, 2]$

   (c) $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0, 8]$

   (d) $y = e^{-x}$, $[0, 1]$

   (e) $y = \sin x$, $[0, \pi]$

   (f) Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ about the x-axis.

   (g) Show that the surface area of a right circular cone of radius r and height h is $\pi r\sqrt{r^2 + h^2}$.
   Hint: Rotate a line $y = mx$ about the x-axis for $0 \leq x \leq h$, where m is determined by the radius r.

MA 114 Worksheet 21: Centers of Mass

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$.

2. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3, 0)$, $(b, 0)$, and $(0, 6)$, where $b > 3$. Find the center of mass.

3. Consider the region under the graph of $y = 1 - x^2$ for $0 \leq x \leq 1$.

   (a) Given uniform density $\rho = 1$, compute the mass M and the moments M_x and M_y.

   (b) Given uniform density $\rho = 2$, compute the mass M and the moments M_x and M_y.

   (c) Use the previous information to compute the centroid of the region in both cases.

   (d) Does the centroid change if ρ changes?

4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.

5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $3/2 \leq x \leq 2$.

MA 114 Worksheet 22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?

   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?

   (d) Represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{c^2} = 1$ with parametric equations.

   (e) Do the two sets of parametric equations

   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$

   and

   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$

   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.

   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?

   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.

   (a) $x = \sqrt{t}$, $y = 1 - t$ for $t \ge 0$.

   (b) $x = 3t - 5$, $y = 2t + 1$ for $t \in \mathbb{R}$.

   (c) $x = \cos(t)$, $y = \sin(t)$ for $t \in [0, 2\pi]$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.

   (a) $y = x^3$ from $x = 0$ to $x = 2$.

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

MA 114 Worksheet 23: Review for Exam 3

1. Find the volume of the following solids.
   (a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the x-axis,
   (b) The solid obtained by rotating the region bounded by $x = y^2$ and $x = 1$ about the line $x = 1$,
   (c) The solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about the line $x = 1$,
   (d) The solid with circular base of radius 1 and cross-sections perpendicular to the base that are equilateral triangles.

2. Find the area of the surface of revolution obtained by rotating the given curve about the given axis.
   (a) $y = \sqrt{x+1},\ 0 \le x \le 3$; about x-axis,
   (b) $x = 3t^2,\ y = 2t^3,\ 0 \le t \le 5$; about y-axis.

3. Compute the arc length of the following curves.
   (a) $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le 2\pi$,
   (b) $y = \sqrt{2 - x^2},\ 0 \le x \le 1$.

4. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.

5. Find the average value of the function $y = 3\sin(x) + \cos(2x)$ on the interval $[0, \pi]$.

6. Compute the slope of the tangent line to the curve in Problem 3(a) above, with $a = 8$, at the point $(1, 3^{3/2})$. Use this to determine an equation for the tangent line.

7. Consider the curve given by the parametric equations $(x(t), y(t)) = (t^2, 2t + 1)$.
   (a) Find the tangent line to the curve at $(4, -3)$. Put your answer in the form $y = mx + b$.
   (b) Find second derivative $\dfrac{d^2y}{dx^2}$ at $(x, y) = (4, -3)$. Is the curve concave up or concave down near this point?