MA 114 Worksheet #09: Recursive Sequences and Series

1. Write out the first five terms of
   (a) $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 3a_{n-1} + {a_n}^2$.
   (b) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$.
   (c) $a_1 = 2$, $a_2 = 1$, and $a_{n+1} = a_n - a_{n-1}$.

2. (a) For what values of x does the sequence $\{x^n\}_{n=1}^{\infty}$ converge?
   (b) For what values of x does the sequence $\{n^x\}_{n=1}^{\infty}$ converge?
   (c) If $\lim\limits_{n\to\infty} b_n = \sqrt{2}$, find $\lim\limits_{n\to\infty} b_{n-3}$.

3. Determine if each of the following sequences is convergent. If the sequence is convergent, find the limit.
   (a) $a_n = 1 + (-1)^n + 2^{-n}$
   (b) $b_n = \dfrac{2^n}{2^n - 1}$
   (c) $c_n = \dfrac{1}{2 + \sin(1/n)}$

4. Newton's method for solving the function $f(x) = x^2 - 2 = 0$ gives a recursive sequence $a_{n+1} = \frac{1}{2}(a_n + 2/a_n)$. Suppose that $\lim_{n\to\infty} a_n = A$ exists. Find an equation satisfied by A. What are the possible values of A.

5. (a) Determine whether the sequence defined as follows is convergent or divergent:
$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n > 1.$$
   (b) What happens if the first term is $a_1 = 2$?

6. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
   (a) Show that the catfish population P_n after n months is given recursively by
$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000.$$
   (b) How many catfish are in the pond after six months?

MA 114 Worksheet #10: Series and The Integral Test

1. Identify the following statements as true or false and explain your answers.
   (a) If the sequence of partial sums of an infinite series is bounded the series converges.
   (b) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_n$ if the series converges.
   (c) $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$ if both series converge.
   (d) If c is a nonzero constant and if $\sum_{n=1}^{\infty} ca_n$ converges then so does $\sum_{n=1}^{\infty} a_n$.
   (e) A finite number of terms of an infinite series may be changed without affecting whether or not the series converges.
   (f) Every infinite series with only finitely many nonzero terms converges.
2. Write the following in summation notation:
   (a) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \dots$
   (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$
3. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series $S = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$.
4. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:
   (a) $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \dots$
   (b) $\sum_{n=0}^{\infty} \left(\frac{\pi}{e} \right)^n$
5. Use the Integral Test to determine if the following series converge or diverge:
   (a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$
   (b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$
   (c) $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$
6. Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise by Integral Test.

MA 114 Worksheet #11: Comparison and Limit Comparison Tests

1. (a) Explain the test for divergence. Why should you never use this test to prove that a series converges?

   (b) State the comparison test for series. Explain the idea behind this test.

   (c) Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy $0 \leq x_n \leq y_n$ for all n and that $\sum_{n=1}^{\infty} y_n$ is convergent. What can you conclude? What can you conclude if instead $\sum_{n=1}^{\infty} y_n$ diverges?

   (d) State the limit comparison test. Explain how you apply this test.

2. Use the appropriate test — Divergence Test, Comparison Test or Limit Comparison Test — to determine whether the infinite series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2+5^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^4}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$

   (g) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{10}{n}\right)^{10}$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (j) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+n+1}{3n^2+14n+7}$

   (k) $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{2+5^n}$

   (l) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^2+5n+2}$

   (m) $\displaystyle\sum_{n=0}^{\infty} \frac{e^{1/n}}{n}$

   (n) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2-\cos^2 n}$

MA 114 Worksheet #12: Alternating Series, Absolute Convergence, & Conditional Convergence

1. (a) Let $a_n = \dfrac{n}{3n+1}$. Does $\{a_n\}$ converge? Does $\sum_{n=1}^{\infty} a_n$ converge?
   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} a_n = 0$.
   (c) Does there exist a convergent series $\sum_{n=1}^{\infty} a_n$ which satisfies $\lim_{n\to\infty} a_n \neq 0$? Explain.
   (d) When does a series converge absolutely? When does a series converge conditionally?
   (e) State the alternating series test.
   (f) Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
   (g) State the Alternating Series Estimation Theorem.

2. Test the following series for convergence or divergence.
   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$
   (b) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{2/3}}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n + 5^n}$
   (e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-5}{18}\right)$

3. Use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

4. Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$ correct to four decimal places; *i. e.* so that $|\text{error}| < 0.00005$.

MA 114 Worksheet #13: The Ratio and Root Tests

1. (a) State the Root Test.
   (b) State the Ratio Test.

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$
   (c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^3+2n}{4n^3+1}\right)^n$
   (d) $\displaystyle\sum_{n=1}^{\infty} 13\cos(5)^{n-1}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$

3. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.
   (a) To prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges you should compute the limit $\displaystyle\lim_{n\to\infty} a_n$. If this limit is 0 then the series converges.
   (b) To apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If this limit is less than 1 then the series converges absolutely.
   (c) To apply the Root Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is 1 or larger then the series diverges.
   (d) One way to prove that a series is convergent is to prove that it is absolutely convergent.
   (e) An infinite series converges when the sequence of partial sums converges.

MA 114 Worksheet #14: Power Series

1. (a) Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$

   (b) For what values of x does the series $\sum_{n=1}^{\infty} 2\big(\cos(x)\big)^{n-1}$ converge?

   (c) Find a formula for the coefficients c_k of the power series $\frac{1}{0!}+\frac{2}{1!}x+\frac{3}{2!}x^2+\frac{4}{3!}x^3+\cdots$.

   (d) Find a formula for the coefficients c_n of the power series $1+2x+x^2+2x^3+x^4+2x^5+x^6+\cdots$.

   (e) Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$ where $c \neq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.

   (f) Consider the function $f(x) = \dfrac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every x satisfying $|x| < 1$.

   (g) Define the terms *power series*, *radius of convergence*, and *interval of convergence.*

2. Find the radius and interval of convergence for

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$.

   (b) $\displaystyle 4\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$.

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$.

   (d) $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^n$.

   (e) $\displaystyle\sum_{n=0}^{\infty} (5x)^n$

   (f) $\displaystyle\sum_{n=0}^{\infty} \sqrt{n}x^n$

   (g) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}}$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{3^n \ln n}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^n}$

   (j) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^4}$

   (k) $\displaystyle\sum_{n=0}^{\infty} \frac{(5x)^n}{n^3}$

3. Use term-by-term integration and the fact that $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$ to derive a power series centered at $x = 0$ for the arctangent function. HINT: $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)}$.

4. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\displaystyle\int_0^x \frac{1}{1+t}\,dt = \ln(1+x)$. You will again want to manipulate the fraction $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)}$ as above.

5. Write $(1 + x^2)^{-2}$ as a power series. HINT: use term-by-term differentiation.

MA 114 Worksheet #15: Taylor and Maclaurin Series

1. (a) Suppose that $f(x)$ has a power series representation for $|x| < R$. What is the general formula for the Maclaurin series for f?

   (b) Suppose that $f(x)$ has a power series representation for $|x - a| < R$. What is the general formula for the Taylor series for f about a?

   (c) Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$. Find the Maclaurin series for f.

   (d) Let $f(x) = 1 + 2x + 3x^2 + 4x^3$. Find the Taylor series for $f(x)$ centered at $x = 1$.

2. Assume that each of the following functions has a power series expansion. Find the Maclaurin series for each. Be sure to provide the domain on which the expansion is valid.

   (a) $f(x) = \ln(1 + x)$

   (b) $f(x) = xe^{2x}$

3. Use a known Maclaurin series to obtain the Maclaurin series for the given function. Specify the radius of convergence for the series.

   (a) $f(x) = \dfrac{x^2}{1 - 3x}$

   (b) $f(x) = e^x + e^{-x}$

   (c) $f(x) = e^{-x^2}$

   (d) $f(x) = x^5 \sin(3x^2)$

   (e) $f(x) = \sin^2 x$.
   HINT: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

4. Find the following Taylor expansions about $x = a$ for each of the following functions and their associated radii of convergence.

   (a) $f(x) = e^{5x}$, $a = 0$.

   (b) $f(x) = \sin(\pi x)$, $a = 1$.

5. Differentiate the series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

   to find a Taylor series for $\cos(x)$.

6. Use Maclaurin series to find the following limit: $\displaystyle\lim_{x \to 0} \frac{x - \tan^{-1}(x)}{x^3}$.

7. Approximate the following integral using a 6th order Taylor polynomial for $\cos(x)$: $\displaystyle\int_0^1 x \cos(x^3)\, dx$

8. Use power series multiplication to find the first three terms of the Maclaurin series for $f(x) = e^x \ln(1 - x)$.

MA 114 Worksheet #16: Review for Exam 02

1. List the first five terms of the sequence:

   (a) $a_n = \dfrac{(-1)^n n}{n! + 1}$

   (b) $a_1 = 6,\ a_{n+1} = \dfrac{a_n}{n}$.

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = 3^n 7^{-n}$

   (b) $a_n = \dfrac{(-1)^{n+1} n}{n + \sqrt{n}}$

   (c) $a_n = \dfrac{\ln n}{\ln 2n}$

   (d) $a_n = \dfrac{\cos^2 n}{2^n}$

3. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 2$.

4. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{n-1}}{3^n}$

5. Determine whether the given series converges or diverges and state which test you used.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{7\sqrt{n}}{5n^{3/2} + 3n - 2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n! e^{-8n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{5n + 7}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{9n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

6. Determine whether the given series is absolutely convergent or conditionally convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n + 1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

7. Find the radius and interval of convergence of the series.

(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ (c) $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

8. Find a power series representation for the function and determine its radius of convergence.

(a) $f(x) = \frac{5}{1-4x^2}$

(b) $f(x) = \frac{x^2}{x^4+16}$

(c) $f(x) = \frac{3}{2+2x}$

(d) $f(x) = e^{-x^2}$

9. Using the formula

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

find a power series for $\ln(1+x)$ and state its radius of convergence.

10. Use the Maclaurin series for $\cos(x)$ to compute

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2}.$$