MA 114 Worksheet #17: Average value of a function

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$

   (b) $f(x) = x^3,\ [-1, 1]$

   (c) $f(x) = \cos(x),\ \left[0, \frac{\pi}{6}\right]$

   (d) $f(x) = \dfrac{1}{x^2+1},\ [-1, 1]$

   (e) $f(x) = \dfrac{\sin(\pi/x)}{x^2},\ [1, 2]$

   (f) $f(x) = e^{-nx},\ [-1, 1]$

   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$

   (h) $f(x) = x^n$ for $n \geq 0$, $[0, 1]$

3. In a certain city the temperature (in $^\circ F$) t hours after 9 am was modeled by the function $T(t) = 50 + 14 \sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9 am to 9 pm.

4. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

   where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity (with respect to r) over the interval $0 < r < R$. Compare the average velocity with the maximum velocity.

5. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t. Then use this new function to compute the average volume of inhaled air in the lungs in one respiratory cycle.

MA 114 Worksheet #18: Volumes I

1. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

2. Calculate the volume of the solid: the base is a square, one of whose sides is the interval $[0, l]$ along the x-axis, and the cross sections perpendicular to the x-axis are rectangles of height $f(x) = x^2$.

3. Calculate the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$, where cross sections perpendicular to the y-axis are squares.

4. The base of a certain solid is the triangle with vertices at $(10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

5. Calculate the volume of the solid whose base is a circle of radius r centered at the origin and which has square cross sections perpendicular to the x-axis.

6. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \leq y \leq 4\}$. The cross sections perpendicular to the y-axis are right isosceles triangles whose hypotenuse lies in the region.

7. Sketch the solid whose volume is given by the integral
$$\pi \int_0^1 (y^2 + 1)^2 - 1\, dy.$$

8. In parts a) and b), use disks or washers to find the an integral expression for the volume of the solid obtained by rotating the given region about the specified line. Evaluate the integrals the integrals to find the volume.
   (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x-axis.
   (b) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.
   (c) Compare the volumes you found in parts (a) and (b). Which is bigger? Why?

9. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y-axis.

10. The torus is the solid obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the y-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]

MA 114 Worksheet #19: Volumes II

1. (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y-axis using the method of cylindrical shells.

   (b) If you use the disk method to compute the same volume, are you integrating with respect to x or y? Why?

2. Sketch the enclosed region and use the shell method to calculate the volume of the solid obtained by rotating the region about the y-axis.

   (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$

   (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$

   (c) $y = 8 - x^3$, $y = 8 - 4x$, for $x \geq 0$

3. For each of the following, sketch the region R and write an integral to give the volume of the solid obtained by rotating R about the specified line.

   (a) R is region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.

   (b) R is the region bounded by $y = 2 - x^2$ and $y = 1$; about the line $y = -1$.

   (c) R is region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

4. Consider the region R which is enclosed by the curves $y = 2x$ and $y = x^2$. The region R is rotated about the x-axis to obtain a solid S.

   (a) Express the volume of S as an integral using the method of washers.

   (b) Express the volume of S as an integral using the method of shells.

   (c) Evaluate the integrals you found in parts a) and b). Check your work by comparing your answers.

5. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the y-axis. Soda is extracted from the glass through a straw at the rate of 1/2 cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

MA 114 Worksheet #20: Arc length and surface area

1. (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.

   (b) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the x-axis. Include the required conditions on $f(x)$.

2. Find an integral expression for the arc length of the following curves. Do ***not*** evaluate the integrals.

   (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$.

   (b) $f(x) = x^4$ from $x = 2$ to $x = 6$.

   (c) $x^2 + y^2 = 1$

3. Find the arc length of the following curves.

   (a) $f(x) = x^{3/2}$ from $x = 100$ to $x = 101$.

   (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$.

   (c) $f(x) = e^x$ from $x = 0$ to $x = 1$.

4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.

5. Compute the surface areas of revolution about the x-axis over the given interval for the following functions.

   (a) $y = x$, $[0, 4]$

   (b) $y = x^3$, $[0, 2]$

   (c) $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0, 8]$

   (d) $y = e^{-x}$, $[0, 1]$

   (e) $y = \sin x$, $[0, \pi]$

   (f) Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ about the x-axis.

   (g) Show that the surface area of a right circular cone of radius r and height h is $\pi r\sqrt{r^2 + h^2}$.
   Hint: Rotate a line $y = mx$ about the x-axis for $0 \leq x \leq h$, where m is determined by the radius r.

MA 114 Worksheet #21: Centers of mass and moments

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$.
2. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3, 0)$, $(b, 0)$, and $(0, 6)$, where $b > 3$. Find the center of mass.
3. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \leq x \leq 1$.
4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.
5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $3/2 \leq x \leq 2$.

MA 114 Worksheet #22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?

   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?

   (d) Represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1$ with parametric equations.

   (e) Do the two sets of parametric equations

   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$

   and

   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$

   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.

   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?

   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.

   (a) $x = \sqrt{t}$, $y = 1 - t$ for $t \ge 0$.

   (b) $x = 3t - 5$, $y = 2t + 1$ for $t \in \mathbf{R}$.

   (c) $x = \cos(t)$, $y = \sin(t)$ for $t \in [0, 2\pi]$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.

   (a) $y = x^3$ from $x = 0$ to $x = 2$.

   (b) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

6. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.
   (b) $x = \cos(\theta) + \sin(2\theta)$, $y = \cos(\theta)$, at $\theta = \pi/2$.

7. For the following parametric curve, find dy/dx.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.

8. Find d^2y/dx^2 for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \dfrac{1}{t}$, $0 < t \leq \pi$.

9. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.

10. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius r centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define θ to be the angle formed by the x-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
    (a) Draw a picture and label θ.
    (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$. Hint: For any angle θ, the line segment between the point up to which you have unwrapped the string and the point at the end of the string has the same length as the arc of the circle corresponding to the angle θ.
    (c) Find the length of the involute for $0 \leq \theta \leq 2\pi$.

MA 114 Worksheet #23: Polar coordinates

1. Convert from rectangular to polar coordinates:
   (a) $(1, \sqrt{3})$
   (b) $(-1, 0)$
   (c) $(2, -2)$

2. Convert from polar to rectangular coordinates:
   (a) $\left(2, \frac{\pi}{6}\right)$
   (b) $\left(-1, \frac{\pi}{2}\right)$
   (c) $\left(1, -\frac{\pi}{4}\right)$

3. List all the possible polar coordinates for the point whose polar coordinates are $(-2, \pi/2)$.

4. Sketch the graph of the polar curves:
   (a) $\theta = \frac{3\pi}{4}$
   (b) $r = \pi$
   (c) $r = \cos\theta$
   (d) $r = \cos(2\theta)$
   (e) $r = 1 + \cos\theta$
   (f) $r = 2 - 5\sin\theta$

5. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{3}$.

6. Find the polar equation for:
   (a) $x^2 + y^2 = 9$
   (b) $x = 4$
   (c) $y = 4$
   (d) $xy = 4$

7. Convert the equation of the circle $r = 2\sin\theta$ to rectangular coordinates and find the center and radius of the circle.

8. Find the distance between the polar points $(3, \pi/3)$ and $(6, 7\pi/6)$.

MA 114 Worksheet #24: Review for Exam 03

1. Find the volume of the following solids.
   (a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the x-axis,
   (b) The solid obtained by rotating the region bounded by $x = y^2$ and $x = 1$ about the line $x = 1$,
   (c) The solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about the line $x = 1$,
   (d) The solid with circular base of radius 1 and cross-sections perpendicular to the base that are equilateral triangles.
2. Find the area of the surface of revolution obtained by rotating the given curve about the given axis.
   (a) $y = \sqrt{x+1},\ 0 \leq x \leq 3$; about x-axis,
   (b) $x = 3t^2,\ y = 2t^3,\ 0 \leq t \leq 5$; about y-axis.
3. Compute the arc length of the following curves.
   (a) $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \leq \theta \leq 2\pi$,
   (b) $y = \sqrt{2 - x^2},\ 0 \leq x \leq 1$.
4. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.
5. Find the average value of the function $y = 3\sin(x) + \cos(2x)$ on the interval $[0, \pi]$.
6. Compute the slope of the tangent line to the curve in Problem 3(a) above, with $a = 8$, at the point $(1, 3^{3/2})$. Use this to determine an equation for the tangent line.
7. Consider the curve given by the parametric equations $(x(t), y(t)) = (t^2, 2t + 1)$.
   (a) Find the tangent line to the curve at $(4, -3)$. Put your answer in the form $y = mx + b$.
   (b) Find second derivative $\dfrac{d^2y}{dx^2}$ at $(x, y) = (4, -3)$. Is the curve concave up or concave down near this point?