MA 114 Worksheet #25: Calculus with polar coordinates

1. Find dy/dx for the following polar curves.

   (a) $r = 2\cos\theta + 1$ (b) $r = 1/\theta$ (c) $r = 2e^{-\theta}$

2. In each of the following, compute the slope of the tangent line at the given point. Then sketch the curve and the tangent line.

   (a) $r = \sin\theta$ at $\theta = \pi/3$. (b) $r = 1/\theta$ at $\theta = \pi/2$.

3. (a) Give the formula for the area of region bounded by the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$. Give a geometric explanation of this formula.

   (b) Give the formula for the length of the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$.

   (c) Use these formulas to establish the formulas for the area and circumference of a circle.

4. Find the slope of the tangent line to the polar curve $r = \theta^2$ at $\theta = \pi$.

5. Find the point(s) where the tangent line to the polar curve $r = 2 + \sin\theta$ is horizontal.

6. Find the area enclosed by one leaf of the curve $r = \sin 2\theta$.

7. Find the arc length of the curve $r = 5^\theta$ for $\theta = 0$ to $\theta = 2\pi$.

8. Find the area of the region bounded by $r = \cos\theta$ for $\theta = 0$ to $\theta = \pi/4$.

9. Find the area of the region that lies inside both the curves $r = \sqrt{3}\sin\theta$ and $r = \cos\theta$.

10. Find the area in the first quadrant that lies inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$.

11. Find the length of the curve $r = \theta^2$ for $0 \le \theta \le 2\pi$.

12. Write down an integral expression for the length of the curve $r = \sin\theta + \theta$ for $0 \le \theta \le \pi$ but do not compute the integral.

MA 114 Worksheet #26: Conic Sections

1. The point in a lunar orbit nearest the surface of the moon is called perilune and the point farthest from the surface is called apolune. The Apollo 11 spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

2. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

3. Use parametric equations and Simpson's Rule with $n = 4$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$. Hint: Find the arc-length of the portion of the ellipse that lies in the first quadrant using Simpson's rule and then multiple by four to obtain the total circumference.

4. Find the area of the region enclosed by the hyperbola $4x^2 - 25y^2 = 100$ and the vertical line through a focus.

5. If an ellipse is rotated about its major axis, find the volume of the resulting solid.

6. Find the centroid of the region enclosed by the x-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

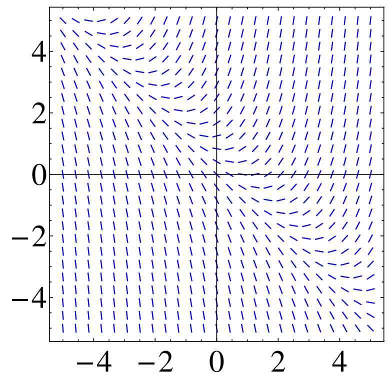
MA 114 Worksheet #27: Differential equations

1. (a) Is $y = \sin(3x) + 2e^{4x}$ a solution to the differential equation $y'' + 9y = 50e^{4x}$? Explain why or why not.
   (b) Explain why every solution of $dy/dx = y^2 + 6$ must be an increasing function.
   (c) What does is mean to say that a differential equation is linear or nonlinear?
2. Find all values of α so that $y(x) = e^{\alpha x}$ is a solution of the differential equation $y'' + y' - 12y = 0$.
3. Find the solution of the initial value problem
$$y(0) = 2, \qquad y' = 3 - y.$$
4. Find the solution of the initial value problem
$$y(0) = -2, \qquad y' = y^2.$$
5. Consider a solution of the differential equation $y' = 3y - 2$. For which values of y is the solution increasing? For which values of y is the solution decreasing?
6. A tank has pure water flowing into it at 10 liters/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liters/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 liters of water. Formulate an initial value problem (that is, a differential equation along with initial conditions) whose solution is the quantity of salt in the tank at any time t. Do not solve the initial value problem.
7. Consider a tank with 200 liters of salt-water solution. A salt-water solution, with a concentration of 2 grams per liter, pours into the tank at a rate of 4 liters per minute. The well-mixed solution in the tank pours out at the same rate of 4 liters/minute. Write a differential equation expressing the rate of change in the concentration, $c(t)$, of salt in the tank. Do not solve.

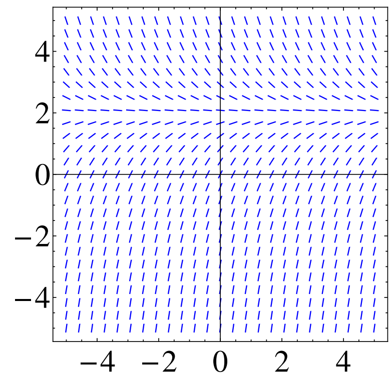
MA 114 Worksheet #28: Direction fields, Separable differential equations

1. Match the differential equation with its slope field. Give reasons for your answer.

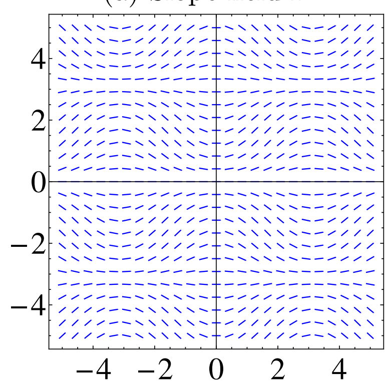
$$y' = 2 - y \qquad y' = x(2-y) \qquad y' = x + y - 1 \qquad y' = \sin(x)\sin(y)$$



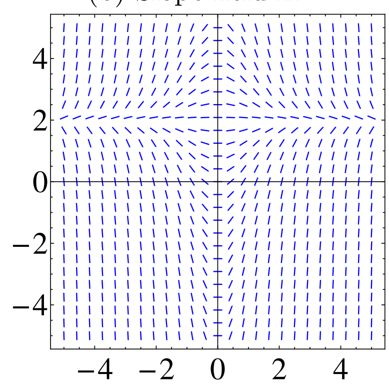
(a) Slope field I



(b) Slope field II



(c) Slope Field III



(d) Slope field IV

Figure 1: Slope fields for Problem 1

2. Use slope field labeled IV to sketch the graphs of the solutions that satisfy the given initial conditions.

$$y(0) = -1, \quad y(0) = 0, \quad y(0) = 1.$$

3. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.
   (a) $y' = y^2$, $(1, 1)$
   (b) $y' = y - 2x$, $(1, 0)$
   (c) $y' = xy - x^2$, $(0, 1)$

4. Consider the autonomous (depends only on y and its derivatives) differential equation $y' = y^2(3-y)(y+1)$. Without solving the differential equation, determine the value of $\lim_{t\to\infty} y(t)$, where the initial value is
   (a) $y(0) = 1$,
   (b) $y(0) = 4$,
   (c) $y(0) = -4$.
5. Use Euler's method with step size 0.5 to compute the approximate y-values, y_1, y_2, y_3, and y_4 of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.
6. Use separation of variables to find the general solutions to the following differential equations.
   (a) $y' + 4xy^2 = 0$
   (b) $\sqrt{1-x^2}y' = xy$
   (c) $(1+x^2)y' = x^3y$
   (d) $y' = 3y - y^2$

MA 114 Worksheet #29: Review for Exam 04

This review worksheet covers only material discussed since Exam III.

To review for your final exam, be sure to study the material from Exams I, II, and III and the review sheets for these exams.

1. Identify and graph the conic section given by each of the following equations. Where applicable, find the foci.
   (a) $x^2 = 4y - 2y^2$
   (b) $x^2 + 3y^2 + 2x - 12y + 10 = 0$

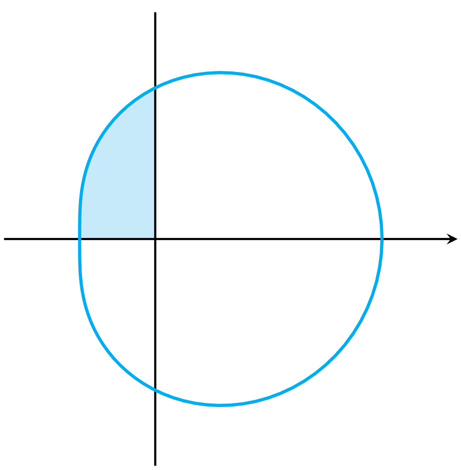
2. Solve the initial value problem
$$\frac{dy}{dx} = \frac{x \sin x}{y}, \quad y(0) = -1$$
   It will help to know that
$$\int x \sin x \, dx = \sin x - x \cos x + C$$

3. By converting to Cartesian coordinates, identify and graph the curve $r^2 \sin 2\theta = 1$ (It may help to remember the identity $\sin 2\theta = 2 \sin\theta \cos\theta$).

4. Draw a direction field for the differential equation $y' = y(1-y)$. What are the equilibria? Classify each as stable or unstable.

5. Find the slope of the tangent line to the curve $r = 2\cos\theta$ at $\theta = \pi/3$.

6. Find the area of the region shown.



$r = 2 + \cos\theta$

7. Find the exact length of the polar curve $r = \theta^2$ for $0 \leq \theta \leq 2\pi$.

8. Use Euler's method with step size 0.1 to estimate $y(0.5)$, where $y(x)$ is the solution of the initial-value problem $y' = y + xy$, $y(0) = 1$.

9. Use Euler's method with step size 0.2 to estimate $y(1)$, where $y(x)$ is the solution of the initial-value problem $y' = x^2 y - \frac{1}{2}y^2$, $y(0) = 1$.

10. Solve the following differential equations.
    (a) $\dfrac{dy}{dx} = 3x^2y^2$
    (b) $xyy' = x^2 + 1$
    (c) $\dfrac{dy}{dx} + e^{x+y} = 0$

11. (a) Solve the differential equation $y' = 2x\sqrt{1 - y^2}$.
    (b) Solve the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 0$, and graph the solution.
    (c) Does the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$, have a solution? Explain.