Exam 1

Name: ______________________________ Section and/or TA: ________________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 A B C D E **6** A B C D E

2 A B C D E **7** A B C D E

3 A B C D E **8** A B C D E

4 A B C D E **9** A B C D E

5 A B C D E **10** A B C D E

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. If $\mathbf{a} = \langle 1, 0, 1\rangle$ and $\mathbf{b} = \langle 2, 3, 2\rangle$ then $|2\mathbf{a} + \mathbf{b}| =$
   - A. $\sqrt{27}$
   - B. $\sqrt{86}$
   - C. $\sqrt{54}$
   - D. $\langle 4, 3, 4\rangle$
   - **E.** $\sqrt{41}$

2. The sphere with equation

$$x^2 + y^2 + z^2 - 2x - 4y + 8z = 15$$

   has
   - A. Center $(-1, -2, -4)$ and radius 6
   - **B. Center $(1, 2, -4)$ and radius 6**
   - C. Center $(1, 4, 3)$ and radius 6
   - D. Center $(-1, -4, -3)$ and radius 6
   - E. Center $(2, 4, -8)$ and radius 12

3. Find the scalar and vector projections of $\mathbf{b} = \langle 4, 6\rangle$ onto $\mathbf{a} = \langle -5, 12\rangle$
   - **A. Scalar projection 4, vector projection $\langle -\frac{20}{13}, \frac{48}{13}\rangle$**
   - B. Scalar projection -4, vector projection $\langle \frac{20}{13}, \frac{-48}{13}\rangle$
   - C. Scalar projection $\sqrt{52}$, vector projection $\langle -\frac{5\sqrt{52}}{13}, \frac{12\sqrt{52}}{13}\rangle$
   - D. Scalar projection $-\sqrt{52}$, vector projection $\langle \frac{5\sqrt{52}}{13}, -\frac{12\sqrt{52}}{13}\rangle$
   - E. Scalar projection 52, vector projection $\langle 260, 624\rangle$

4. Find the volume of the parallelepiped determined by the vectors $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle -1, 1, 2 \rangle$, and $\mathbf{c} = \langle 2, 1, 4 \rangle$

   A. 3

   B. 6

   C. 9

   D. -9

   E. 12

5. Which of the following best describes the graph of the equation $x^2 - y^2 + z^2 = 1$?

   A. A sphere of radius 1

   B. An elliptic cylinder

   C. A hyperboloid of one sheet with the x axis as an axis of symmetry

   D. A hyperboloid of one sheet with the y axis as axis of symmetry

   E. A hyperboloid of one sheet with the z axis as axis of symmetry

6. The function $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j} + \mathbf{k}$ traces out:

   A. A circle of radius 5 in the xy plane

   B. An ellipse in the plane $z = 1$

   C. An ellipse in the plane $x = 3$

   D. An ellipse in the plane $y = 4$

   E. A circle of radius 5 in the plane $z = 1$

7. The curves $\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}_2(t) = \langle \sin t, \sin 2t, t \rangle$ intersect at $(0,0,0)$ where $t = 0$. What is the angle of intersection?

   A. $\cos^{-1}(5/\sqrt{6})$

   B. $\cos^{-1}(1/\sqrt{6})$

   C. 0

   D. $\cos^{-1}(\sqrt{6})$

   E. $\sin^{-1}(\sqrt{6})$

8. The curve $\mathbf{r}(t) = \langle t\cos t, t\sin t, t \rangle$ lies on which one of the following surfaces?

   A. $z^2 = x^2 + y^2$

   B. $z = x^2 + y^2$

   C. $z^2 = x^2 - y^2$

   D. $z = x^2 - y^2$

   E. $x = y$

9. Consider the planes given by the equations

$$\begin{aligned} x + 2y - z &= 0 \\ -4x + 3y + 2z &= 1 \end{aligned}$$

   Which one of the following statements is correct?

   A. These planes are parallel

   B. These planes are neither parallel nor do they intersect

   C. These planes are perpendicular and the vector $\mathbf{v} = \langle 7, 2, 11 \rangle$ points along the line of intersection.

   D. These planes intersect only in the point $(0, 1/7, 2/7)$

   E. These planes intersect only in the point $(-1/2, 0, 1/2)$

10. Consider the triangle with vertices $P(3,-2,3)$, $Q(7,0,1)$, $R(1,2,1)$. Which one of the following statements is correct?

    A. ΔPQR is an equilateral triangle

    B. ΔPQR is an isosceles triangle

    C. ΔPQR is a right triangle, but not isosceles

    D. ΔPQR has no equal sides

    E. None of the above

Free Response Questions

11. (10 points) Find the equation of the plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$. Express your answer in the form $ax + by + cz = d$

Solution:

Normals to the given planes are

$$\mathbf{n}_1 = \langle 2, 1, -2 \rangle \quad \text{(2 points)}$$
$$\mathbf{n}_2 = \langle 1, 0, 3 \rangle \quad \text{(2 points)}$$

so a normal to the plane perpendicular to both of these is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 8\mathbf{j} - \mathbf{k} \quad \text{(3 points)}$$

Hence, the equation of the plane is

$$3(x-1) - 8(y-5) - (z-1) = 0 \quad \text{(3 points)}$$

or

$$3x - 8y - z = -38 \quad \text{(Students don't have to simplify)}$$

12. (15 points) (a) (7 points) Find the point of intersection between the curves $\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle$ and $\mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$

Solution: Equating the x, y, and z coordinates of the two curves we get the conditions

$$t = 3 - s \qquad \text{(1 points)}$$
$$1 - t = s - 2 \qquad \text{(1 points)}$$
$$3 + t^2 = s^2 \qquad \text{(1 points)}$$

If we substitute $t = 3 - s$ (from the first equation) into the third equation we get

$$3 + (3-s)^2 = s^2$$

or $3 + 9 - 6s = 0$ so $s = 2$. From the first equation, $t = 1$, which is consistent with the second equation. Hence the intersection occurs at $s = 2$, $t = 1$. The coordinates of the intersection are then

$$\mathbf{r}_1(1) = \langle 1, 0, 4 \rangle.$$

Solve simultaneous equations - 3 points

Answer - 1 point

(b) (8 points) The curves

$$\mathbf{r}_1(t) = \langle -2t, t^5, -5t^3 \rangle$$

and

$$\mathbf{r}_2(u) = \langle \sin(-2u), \sin(u), u - \pi \rangle$$

intersect at $\langle 0, 0, 0 \rangle$. Find the values of t and u corresponding to the point $\langle 0, 0, 0 \rangle$ and find the angle of intersection of the two curves at this point. It is OK to leave your answer is the form $\cos^{-1}(\text{angle})$.

Solution: First, find the tangent vectors at $t = 0$ and $u = \pi$

$$\mathbf{r}_1'(t) = \langle -2, 5t^4, -15t^2 \rangle \qquad \text{(1 points)}$$
$$\mathbf{r}_2'(u) = \langle -2\cos(-2u), \cos(u), 1 \rangle \qquad \text{(1 points)}$$

so

$$\mathbf{r}_1'(0) = \langle -2, 0, 0 \rangle \qquad \text{(1 points)}$$
$$\mathbf{r}_2'(\pi) = \langle -2, -1, 1 \rangle \qquad \text{(1 points)}$$

and

$$\left|\mathbf{r}_1'(0)\right| = 2 \qquad \text{(1 points)}$$
$$\left|\mathbf{r}_2'(\pi)\right| = \sqrt{6} \qquad \text{(1 points)}$$

while

$$\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(\pi) = 4 \quad \text{(1 points)}$$

The cosine of the angle of intersection is given by

$$\cos(\theta) = \frac{4}{2 \cdot \sqrt{6}} = \frac{2}{\sqrt{6}} \quad \text{(1 points)}$$

13. (10 points) (a) (5 points) Find a function $\mathbf{r}(t)$ that describes the curve of intersection between the cylinder $x^2 + y^2 = 16$ and the plane $x + z = 5$. *Hint*: $x^2 + y^2 = 16$ is the equation of a circle so x and y can be parameterized using trig functions.

Solution: Set

$$x(t) = 4\cos t, \quad y(t) = 4\sin t. \quad \text{(1 points)}$$

Using the equation of the plane we see that

$$z(t) = 5 - x(t) = 5 - 4\cos(t). \quad \text{(2 points)}$$

Hence the curve of intersection is given by

$$\mathbf{r}(t) = \langle 4\cos t, 4\sin t, 5 - 4\cos t\rangle. \quad \text{(2 points)}$$

(b) (5 points) At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t\rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$? Be sure to give the x, y and z coordinates of each point of intersection.

Solution: Any point of intersection will satisfy the equation

$$\sin^2 t + \cos^2 t + t^2 = 5 \quad \text{(1 points)}$$

or

$$1 + t^2 = 5 \quad \text{(1 points)}$$

so that $t = \pm 2$ (1 points). The two points of intersection are then $(\sin(2), \cos(2), 2)$ and $(\sin(-2), \cos(-2), -2)$. 1 point each

14. (15 points) (a) (6 points) Find the traces of the curve $x^2 + y^2 - z^2 = 1$ in the xy, xz, and yz planes. In each case, identify the conic section.

> **Solution:**
> xy plane: $z = 0$ so $x^2 + y^2 = 1$, a circle of radius 1 (2 points)
> xz plane: $y = 0$ so the trace is $x^2 - z^2 = 1$, a hyperbola (2 points)
> yz plane: $x = 0$ so the trace is $y^2 - z^2 = 1$, a hyperbola (2 points)

(b) (4 points) Find the center of the quadric surface

$$4x^2 + 4y^2 - 8y + z^2 = 0$$

and identify the quadric surface.

> **Solution:** We complete the square to find
>
> $$4x^2 + 4(y-1)^2 + z^2 = 4 \quad \text{(2 points)}$$
>
> so the quadric surface is an ellipsoid with center at $(0, 1, 0)$ (2 points).

(c) (5 points) Using the distance formula, find an equation for the set of all points equidistant from $P(-3, 0, 1)$ and $Q(-1, 2, 1)$. Describe the set (e.g. as a line, plane, sphere, ellipsoid, etc.).

> **Solution:** Suppose $R(x, y, z)$ is such a point, Then $|PR| = |QR|$ or, using the distance formula
>
> $$\begin{aligned} \sqrt{(x+3)^2 + y^2 + (z-1)^2} &= \sqrt{(x+1)^2 + (y-2)^2 + (z-1)^2} \\ x^2 + 6x + 9 + y^2 + (z-1)^2 &= x^2 + 2x + 1 + y^2 - 4y + 4 + (z-1)^2 \\ 6x + 9 &= 2x - 4y + 5 \\ 4x + 4y &= -4 \\ x + y &= -1 \end{aligned}$$
>
> Simpification is optional
>
> This is the equation of a plane perpendicular to the vector $\langle 1, 1, 0 \rangle$ It is also OK to use the vector $\langle 4, 4, 0 \rangle$ or another parallel vector here.
>
> Note: This is to be expected since the vector $\overrightarrow{PQ}$ is $\langle 2, 2, 0 \rangle$.
>
> Scoring:
>
> Use distance formula: 2 points
> Reduce to linear equation: 1 points
>
> Identify the set as a plane: 2 point