Alternate Exam 2

Name: ______________________________ Section and/or TA: ________________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **6** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **7** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **8** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **9** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **10** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. Find the arc length of the curve $r(t) = \langle 2t, 2\cos t, 2\sin t\rangle$ from $(0,2,0)$ to $(\pi,0,2)$.

   A. $\sqrt{5}\pi$

   B. 2π

   C. $\pi\sqrt{2}$

   D. π

   E. $2\sqrt{5}\pi$

2. Find the domain of the function

$$f(x,y,z) = \frac{\ln(4-y)}{\sqrt{x^2+y^2+z^2-25}}$$

   A. $y \neq 4$ and $x^2+y^2+z^2 > 25$

   B. $y > 4$ and $x^2+y^2+z^2 > 25$

   C. $y < 4$ **and** $x^2+y^2+z^2 > 25$

   D. $y \geq 4$ and $x^2+y^2+z^2 < 25$

   E. $x^2+y^2+z^2 \neq 25$ and $y \neq 4$

3. Let

$$u(x,y,z) = xyze^{xyz}.$$

   Find u_y.

   A. e^{xyz}

   B. $(xz+xy+yz)e^{xyz}$

   C. xze^{xyz}

   D. $2xze^{xyz}$

   E. $xz(1+xyz)e^{xyz}$

4. Suppose that z satisfies the equation $xz + x\ln y = z^2$. Assuming that this equation defines z implicitly as a function of (x, y), determine $\partial z/\partial x$ at the point $(x, y, z) = (4, 1, 4)$.

   A. 0

   B. 1/2

   C. 1

   D. 8

   E. $-1/2$

5. Suppose that $z(x, y) = F(u(x, y), v(x, y))$ where F, u and v are differentiable. Suppose that

$$u(1, 2) = 5 \quad v(1, 2) = 2,$$

that

$$u_x(1, 2) = 3, \quad u_y(1, 2) = -1, \quad v_x(1, 2) = -2, \quad v_y(1, 2) = 4$$

and

$$F_u(5, 2) = 6, \quad F_v(5, 2) = -3.$$

Find $z_y(1, 2)$.

   A. 24

   B. 22

   C. -5

   D. -10

   E. -18

6. Find the directional derivative of $f(x, y) = xe^y$ at $(2, 0)$ in the direction of $\mathbf{u} = \langle 3, 3 \rangle$.

   A. $3/\sqrt{2}$

   B. -2

   C. $-1/\sqrt{2}$

   D. $1/\sqrt{2}$

   E. 8

7. Let $f(x,y) = 4 + x^3 + y^3 - 3xy$. Which of the following statements is correct?

   A. f has a saddle point at $(1,1)$

   B. f has a local minimum at $(1,1)$

   C. f has a local maximum at $(1,1)$

   D. f has a global maximum at $(1,1)$

   E. f has neither local maxima, nor local minima, nor saddle points

8. Find the maximum rate of change of the function $f(x,y) = 4y\sqrt{x}$ at the point $(x,y) = (2,2)$.

   A. $\sqrt{20}$

   B. $\sqrt{40}$

   C. 8

   D. 4

   E. 0

9. Find the maximum and minimum values of $f(x,y) = x^2 - xy + y^2$ subject to the constraint $x^2 + y^2 = 1$.

   A. Maximum $3/2$, minimum $1/2$

   B. Maximum $1/\sqrt{2}$, minimum $-1/\sqrt{2}$

   C. Maximum $1/2$, minimum $-1/2$

   D. Maximum $\sqrt{2}$, minimum $-\sqrt{2}$

   E. Maximum 1, Minimum -1

10. Compute the iterated integral

$$\int_0^1 \int_0^1 x^2 y^3 \, dx \, dy$$

   A. $1/12$

   B. $1/6$

   C. $5/6$

   D. $5/12$

   E. $1/3$

Free Response Questions

11. (10 points) The goal of this problem is to determine the maximum and minimum values of the function $f(x,y) = x^2 + 2y^2 - 4y$ on the circle $x^2 + y^2 = 9$. Use the Lagrange multiplier method to obtain your answer. Solutions using other methods will receive <u>*no credit*</u>.

(a) (3 points) Set up the needed partial derivative equations and the constraint equation for the Lagrange multiplier method.

Solution: In the book's notation

$$f(x,y) = x^2 + 2y^2 - 4y$$

and

$$g(x,y) = x^2 + y^2.$$

Since $f_x(x,y) = 2x$ and $f_y(x,y) = 4y - 4$, $g_x(x,y) = 2x$, $g_y(x,y) = 2y$, we get

$$2x = 2\lambda x \qquad \text{(1 points)}$$
$$4y - 4 = 2\lambda y \qquad \text{(1 points)}$$
$$x^2 + y^2 = 9 \qquad \text{(1 points)}$$

(b) (5 points) Determine the critical points as deduced from the Lagrange Multiplier method.

Solution:
From the first Lagrange equation we get $2x(\lambda - 1) = 0$ so either $x = 0$ or $\lambda = 1$. If $x = 0$, the constraint equation gives $y = \pm 3$. If $\lambda = 1$, the second Lagrange equation gives $4y - 4 = 2y$ or $y = 2$, so $x = \pm\sqrt{5}$.

Scoring:

1 point for eliminating λ or equivalent first step
1 point for using the constraint equation to eliminate one of the variables
1 point for solving for x (or y)
2 points for a correct listing of the critical points

(c) (2 points) Identify the absolute maximum and absolute minimum of $f(x,y)$ on the ellipse.

Solution: Evaluating at the critical points we get the following table.

x	y	$f(x,y) = x^2 + 2y^2 - 4y$
0	3	6
0	-3	30
$\sqrt{5}$	2	5
$-\sqrt{5}$	2	5

The absolute maximum is 30 and the absolute minimum is 5.

1 point per correct answer.

0 points total for unsupported answers, even if correct.

12. (15 points) Answer the following questions.

(a) (5 points) Determine an equation of the tangent plane to the surface

$$x^2 + 2y^2 + 3z^2 = 36$$

at the point $(1,2,3)$.

Solution: The gradient of $f(x,y,z) = x^2 + 2y^2 + 3z^2$ at $(1,2,3)$ will be normal to the tangent plane. (1 points)
Since $\nabla f(x,y,z) = \langle 2x, 4y, 6z \rangle$ (1 points), a normal vector is $\langle 2, 8, 18 \rangle$ (1 points), and the equation of the plane takes the form $2x + 8y + 18z = D$.(1 points) Substituting $(x,y,z) = (1,2,3)$, we get $D = 72$ (1 points), so the equation is

$$2x + 8y + 18z = 72$$

or

$$x + 4y + 9z = 36.$$

(1 points)

(b) (5 points) Determine a linear approximation $L(x,y)$ for the function $g(x,y) = xe^{xy}$ at the point $(3,0)$.

Solution: First, note that $g(3,0) = 3$ (1 points).
Next, compute the partial derivatives:

$$g_x(x,y) = e^{xy} + xye^{xy}, \quad g_y(x,y) = x^2e^{xy}, \quad \text{(1 points)}$$

so that

$$g_x(3,0) = 1, \quad g_y(3,0) = 9. \quad \text{(1 points)}$$

Hence,

$$L(x,y) = 3 + (x-3) + 9(y) = x + 9y \quad \text{(2 points)}$$

Students are not required to simplify. Also, it's acceptable for students to find the equation of the tangent plane and solve for z

(c) (5 points) Given that a function $f(x,y)$ is a differentiable function with $f(2,3) = 4$ and $f_x(2,3) = -3$ and $f_y(2,3) = 6$. Use differentials or a linear approximation to estimate $f(1.9, 3.1)$.

Solution: From the given data the linear approximation is

$$L(x,y) = 4 - 3(x-2) + 6(y-3) \quad \text{(3 points)}$$

so

$$f(1.9, 3.1) \simeq L(1.9, 3.1) = 4 - 3(-0.1) + 6(0.1) = 4.9. \quad \text{(2 points)}$$

For $L(x,y)$ (3 points), one point each for the constant term, $x-2$ term, and $y-3$ term.

For $f(1.9, 3.1)$ (2 points), 1 point for the correct expression, and 1 point for a numerically correct answer.

It is also acceptable for students to use differentials $(df = f_x dx + f_y dy)$ to solve this problem.

13. (10 points) (a) (6 points) Suppose that

$$z = u^2 v^2 \quad u = 2s + 3t, \quad v = 3s - 2t.$$

Using the chain rule for functions of several variables, find $\partial z/\partial s$. Express your final answer in terms of s and t. You do not need to simplify. Solutions which do not use the chain rule will receive <u>*no credit*</u>.

Solution: First, note that

$$\frac{\partial z}{\partial u} = 2uv^2 \quad \text{(1 points)} \qquad \frac{\partial z}{\partial v} = 2u^2 v \quad \text{(1 points)}$$

$$\frac{\partial u}{\partial s} = 2 \quad \text{(1 points)} \qquad \frac{\partial v}{\partial s} = 3 \quad \text{(1 points)}$$

Hence

$$\frac{\partial z}{\partial s} = 4uv^2 + 6u^2 v \qquad \text{(1 points)}$$

or

$$\frac{\partial z}{\partial s} = 4(2s+3t)(3s+2t)^2 + 6(2s+3t)(3s-2t)^2 \qquad \text{(1 points)}$$

(b) (4 points) Find $\partial z/\partial x$ and $\partial z/\partial y$ if the equation

$$x^2 + y^2 + z^3 - 2z = 4$$

defines z implicitly as a function of x and y.

Solution: From the given equation we have

$$\begin{aligned} 2x + 3z^2 \frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial x}\left(3z^2 - 2\right) &= -2x \\ \frac{\partial z}{\partial x} &= \frac{-2x}{3z^2 - 2} \end{aligned}$$

(2 points)

In a similar way we get

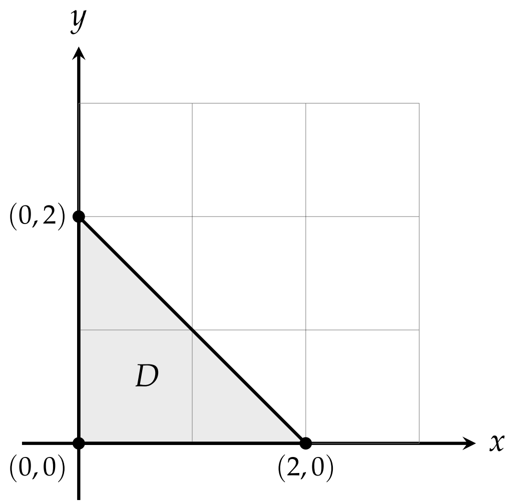
$$\frac{\partial z}{\partial y} = \frac{-2y}{2z^2 - 2}.$$

(2 points)

14. (15 points) The goal of this problem is to find the absolute maximum and minimum values of the function

$$f(x,y) = x^3 - xy + y^2 - x$$

in the closed triangular domain D shown below.



(a) (4 points) Find the critical point(s) of f contained in D and classify the point(s) as local maxima, local minima, or saddles. Find the value of f at the critical point(s).

Solution: Since

$$f_x(x,y) = 3x^2 - y - 1$$
$$f_y(x,y) = -x + 2y$$

setting $x = 2y$ in the first equation gives

$$12y^2 - y - 1 = 0$$

and $y = 1/3$ or $y = -1/4$. Thus there are critical points at $(x,y) = (2/3, 1/3)$ and $(-1/2, -1/4)$. Only the point $(2/3, 1/3)$ lies in D. Since

$$f_{xx}(x,y) = 6x, \quad f_{xy} = -1, \quad f_{yy} = 2$$

we have $f_{xx}(2/3, 1/3) = 4$, $f_{xy}(2/3, 1/3) = -1$, $f_{yy}(x,y) = 2$ and $D = 9$. It follows from the second derivative test that $f(2/3, 1/3) = -13/27$ is a local minimum.

Scoring:

2 point for finding critical points correctly
1 point for evaluating second partials
1 point for using D to identify $(2/3, 1/3)$ as a local minimum
1 point for evaluating f

(b) (3 points) Find the maximum and minimum of $f(x,0)$ for $0 \leq x \leq 2$.

Solution: $g_1(x) = f(x,0) = x^3 - x$ has a critical point at $x = 1/\sqrt{3}$. Evaluate

$g_1(0)$	0
$g_1(1/\sqrt{3})$	$-2/(3\sqrt{3})$
$g_1(2)$	7

The minimum value is $-2/(3\sqrt{3})$ and the maximum value is 7.

Scoring:

1 point for finding the critical point
1 point for correct evaluations at 0, $1/\sqrt{3}$, 2

1 point for identifying the maximum and minimum

(c) (3 points) Find the minimum and maximum value of $f(0,y)$ for $0 \leq y \leq 2$.

Solution: $g_2(y) = f(0,y) = y^2$ has a critical point at $y = 0$ but no interior critical points.

$g_2(0)$	0
$g_2(2)$	4

The minimum value is 0 and the maximum value is 4.
Scoring:

1 point for finding the critical point
1 point for correct evaluations at 0, 1, 2

1 point for identifying the maximum and minimum

(d) (3 points) Find the minimum and maximum values of

$$f(x,2-x) = x^3 + 2x^2 - 7x + 4$$

for $0 \leq x \leq 2$.

Solution:

$$\begin{aligned} g_3(x) &= f(x,2-x) \\ &= x^3 - x(2-x) + (2-x)^2 - x \\ &= x^3 + 2x^2 - 7x + 4. \end{aligned}$$

The derivative is $3x^2 + 4x - 7$ which has zeros at $x = 1$ and $x = -7/3$. We need only consider $x = 1$. Thus:

$g_3(0)$	4
$g_3(1)$	0
$g_3(2)$	6

The minimum value is 0 and the maximum value is 6

Scoring:

1 point for finding the critical point
1 point for correct evaluations at 0, 1, 2

1 point for identifying the maximum and minimum

(e) (2 points) State the maximum and minimum values of f on D.

Solution:
Maximum Value: $-13/27$ (1 points)

Minimum Value: 7 (1 points)