Exam 2

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. Find the arc length of the curve $r(t) = \langle 2t, \cos t, \sin t\rangle$ from $(0,1,0)$ to $(\pi, 0, 1)$.

   A. $\sqrt{5}\pi$

   B. $\sqrt{5}\pi/2$

   C. 2π

   D. π

   E. $2\sqrt{5}\pi$

2. Find the domain of the function

$$f(x,y,z) = \frac{\sqrt{x-1}}{\sqrt{1-y}} + \ln(4 - x^2 - y^2 - z^2)$$

   A. $\{(x,y,z) : y \neq 1\}$

   B. $\{(x,y,z) : x < 1, -2 < y < 1, \text{ and } x^2 + y^2 + z^2 > 4\}$

   C. $\{(x,y,z) : -1 < x < 2 \text{ and } -1 < y < 0\}$

   D. $\{(x,y,z) : 1 < x < 2, -2 < y < 1 \text{ and } x^2 + y^2 + z^2 \leq 4\}$

   E. $\{(x,y,z) : 1 \leq x < 2, -2 < y < 1$ **and** $x^2 + y^2 + z^2 < 4\}$

3. Let

$$u(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

   Find u_y.

   A. $\dfrac{-y}{(x^2+y^2+z^2)^{3/2}}$

   B. $\dfrac{2y}{\sqrt{x^2+y^2+z^2}}$

   C. $\dfrac{y}{\sqrt{x^2+y^2+z^2}}$

   D. $\dfrac{2x+2y+2z}{(x^2+y^2+z^2)^{3/2}}$

   E. $\dfrac{2x+2y+2z}{\sqrt{x^2+y^2+z^2}}$

4. Suppose that z satisfies the equation $yz + x\ln y = z^2$. Assuming that this equation defines z implicitly as a function of (x, y), determine $\partial z/\partial y$ at the point $(x, y, z) = (2, 1, 1)$.

   A. 0

   B. 1

   C. $1/2$

   D. 3

   E. $-1/2$

5. Suppose that $z(x, y) = F(u(x, y), v(x, y))$ where F, u and v are differentiable. Suppose that

$$u(1, 2) = 5 \quad v(1, 2) = 2,$$

   that

$$u_x(1, 2) = 3, \quad u_y(1, 2) = -1, \quad v_x(1, 2) = -2, \quad v_y(1, 2) = 4$$

   and

$$F_u(5, 2) = -2, \quad F_v(5, 2) = 6.$$

   Find $z_y(1, 2)$.

   A. 24

   B. 22

   C. 26

   D. -5

   E. -18

6. Find the directional derivative of $f(x, y) = xe^y$ at $(2, 0)$ in the direction of $\mathbf{u} = \langle 2, -2 \rangle$.

   A. -2

   B. 2

   C. $1/\sqrt{2}$

   D. $-1/\sqrt{2}$

   E. 0

7. Let $f(x,y) = xy - 2x - 2y - x^2 - y^2$. Which of the following statements is correct?

   A. f has a saddle point at $(-2,-2)$

   B. f has a local maximum at $(-2,-2)$

   C. f has a local minimum at $(-2,-2)$

   D. f has a global maximum at $(-2,-2)$

   E. f has neither local maxima, nor local minima, nor saddle points

8. Find the maximum rate of change of the function $f(x,y) = 4y\sqrt{x}$ at the point $(x,y) = (4,1)$.

   A. $\sqrt{17}$

   B. $\sqrt{65}$

   C. 7

   D. 1

   E. 0

9. Find the maximum and minimum values of $f(x,y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 1$.

   A. Maximum 0, minimum -2

   B. Maximum 2, minimum 0

   C. Maximum 2, minimum -2

   D. Maximum $\sqrt{2}$, minimum $-\sqrt{2}$

   E. Maximum 1, Minimum -1

10. Compute the iterated integral

$$\int_0^1 \int_0^1 (x+y)^2 \, dx \, dy$$

   A. $(x+y)^3/3$

   B. $6/7$

   C. $7/6$

   D. $2/3$

   E. $5/6$

Free Response Questions

11. (10 points) The goal of this problem is to determine the maximum and minimum values of the function $f(x,y) = xy$ on the ellipse $4x^2 + y^2 = 8$. Use the Lagrange multiplier method to obtain your answer. Solutions using other methods will receive *no credit*.

(a) (3 points) Set up the needed partial derivative equations and the constraint equation for the Lagrange multiplier method.

Solution: In the book's notation

$$f(x,y) = xy$$

and

$$g(x,y) = 4x^2 + y^2 - 8.$$

Since $f_x(x,y) = y$ and $f_y(x,y) = x$, $g_x(x,y) = 8x$, $g_y(x,y) = 2y$, we get

$$y = 8\lambda x \qquad \text{(1 points)}$$

$$x = 2\lambda y \qquad \text{(1 points)}$$

$$4x^2 + y^2 = 8 \qquad \text{(1 points)}$$

(b) (5 points) Determine the critical points as deduced from the Lagrange Multiplier method.

Solution: From the first equation we get $\lambda = y/8x$. Substituting this into the second equation we get $x = y^2/4x$ or $4x^2 = y^2$. Using the constraint equation we recover $8x^2 = 8$ or $x = \pm 1$. Hence, $y = \pm 2$.

Scoring:

1 point for eliminating λ or equivalent first step
1 point for using the constraint equation to eliminate one of the variables
1 point for solving for x (or y)
2 points for a correct listing of the critical points

(c) (2 points) Identify the absolute maximum and absolute minimum of $f(x,y)$ on the ellipse.

Solution: For the possible critical points we compute

x	**y**	$f(x,y)$
-1	-2	2
1	-2	-2
-1	2	-2
1	2	2

The absolute maximum is 2 and the absolute minimum is -2.

1 point per correct answer.

0 points total for unsupported answers, even if correct.

12. (15 points) Answer the following questions.

(a) (5 points) Determine an equation of the tangent plane to the surface

$$x^2 - y^2 + z^2 - 2z = -3$$

at the point $P = (1, 2, 2)$. Give your answer in the form $Ax + By + Cz = D$.

Solution: The gradient of $f(x, y, z) = x^2 - y^2 + z^2 - 2z$ at $(1, 2, 2)$ will be normal to the tangent plane. (1 points)
Since $\nabla f(x, y, z) = \langle 2x, -2y, 2z - 2\rangle$ (1 points), a normal vector is $\langle 2, -4, 2\rangle$ (1 points), and the equation of the plane takes the form $2x - 4y + 2z = D$.(1 points)
Substituting $(x, y, z) = (1, 2, 2)$, we get $D = -2$ (1 points), so the equation is

$$2x - 4y + 2z = -2$$

or

$$x - 2y + z = 1.$$

(1 points)

(b) (5 points) Determine a linear approximation $L(x, y)$ for the function $g(x, y) = xy^3 + 1$ at the point $(1, 1)$.

Solution: First, note that $g(1, 1) = 2$ (1 points).
Next, compute the partial derivatives:

$$g_x(x, y) = y^3, \quad g_y(x, y) = 3xy^2 \quad \text{(1 points)}$$

so that

$$g_x(1, 1) = 1, \quad g_y(1, 1) = 3. \quad \text{(1 points)}$$

Hence,

$$L(x, y) = 2 + (x - 1) + 3(y - 1) = -2 + x + 3y. \quad \text{(2 points)}$$

Students are not required to simplify. Also, it's acceptable for students to find the equation of the tangent plane and solve for z

(c) (5 points) Given that a function $f(x, y)$ is a differentiable function with $f(2, 3) = 7$ and $f_x(2, 3) = -2$ and $f_y(2, 3) = 5$. Use differentials or a linear approximation to estimate $f(1.9, 3.1)$.

Solution: From the given data the linear approximation is

$$L(x, y) = 7 - 2(x - 2) + 5(y - 3) \quad \text{(3 points)}$$

so

$$f(1.9, 3.1) \simeq L(1.9, 3.1) = 7 - 2(-0.1) + 5(0.1) = 7.7. \quad \text{(2 points)}$$

For $L(x, y)$ (3 points), one point each for the constant term, $x - 2$ term, and $y - 3$ term.

For $f(1.9, 3.1)$ (2 points), 1 point for the correct expression, and 1 point for a numerically correct answer.

It is also acceptable for students to use differentials $(df = f_x dx + f_y dy)$ to solve this problem.

13. (10 points) (a) (6 points) Suppose that

$$z = u/v, \quad u = 2s + 3t, \quad v = 3s - 2t.$$

Using the chain rule for functions of several variables, find $\partial z/\partial s$. Express your final answer in terms of s and t. Solutions which do not use the chain rule will receive <u>*no credit*</u>.

Solution: First, note that

$$\frac{\partial z}{\partial u} = 1/v \quad \text{(1 points)} \qquad \frac{\partial z}{\partial v} = -u/v^2 \quad \text{(1 points)}$$

$$\frac{\partial u}{\partial s} = 2 \quad \text{(1 points)} \qquad \frac{\partial v}{\partial s} = 3 \quad \text{(1 points)}$$

Hence

$$\frac{\partial z}{\partial s} = 2/v - 3u/v^2 \qquad \text{(1 points)}$$

or

$$\frac{\partial z}{\partial s} = \frac{-13t}{(3s - 2t)^2} \qquad \text{(1 points)}$$

(b) (4 points) Find $\partial z/\partial x$ and $\partial z/\partial y$ if the equation

$$x^2 + y^2 + z^2 - 2z = 4$$

defines z implicitly as a function of x and y.

Solution: From the given equation we have

$$2x + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(2z - 2) = -2x$$

$$\frac{\partial z}{\partial x} = -\frac{2x}{2z - 2} = -\frac{x}{z - 1}$$

(2 points) and similarly

$$2y + 2z\frac{\partial z}{\partial y} - 2\frac{\partial z}{\partial y} = 0$$

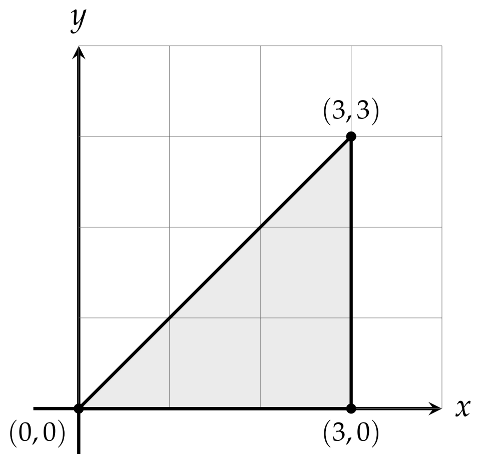
$$\frac{\partial z}{\partial y} = -\frac{2y}{2z - 2} = -\frac{y}{z - 1}$$

(2 points)

14. (15 points) The goal of this problem is to find the absolute maximum and minimum values of the function

$$f(x,y) = x^2 + 4y^2 - 2x - 8y + 10$$

in the closed triangular domain shown at the left.



(a) (4 points) Find the critical point(s) of f and classify the point(s) as local maxima, local minima, or saddles. Be sure to find *all* critical points, whether inside D or outside D. Find the value of f at the critical point(s).

Solution: Since $f_x(x,y) = 2x - 2$ and $f_y(x,y) = 8y - 8$ there is a single critical point at $x = 1$, $y = 1$. We have $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 8$, $f_{xy} = 0$ so $D = 16$ and this critical point is a local minimum. We have $f(1,1) = 5$.

Scoring:

2 point for finding critical points correctly
1 point for evaluating second partials
1 point for using D to identify $(1,1)$ as a local minimum
1 point for evaluating f

(b) (3 points) Find the maximum and minimum of $f(x,0)$ for $0 \le x \le 3$.

Solution: $g_1(x) = f(x,0) = x^2 - 2x + 10$ has a critical point at $x = 1$. Evaluate

$g_1(0)$	10
$g_1(1)$	9
$g_1(3)$	13

The minimum value is 9 and the maximum value is 13.

Scoring:

1 point for finding the critical point

1 point for correct evaluations at 0, 1, 3

1 point for identifying the maximum and minimum

(c) (3 points) Find the minimum and maximum value of $f(3, y)$ for $0 \leq y \leq 3$.

Solution: $g_2(y) = f(3, y) = 4y^2 - 8y + 13$ and has a critical point at $y = 1$. Thus

$g_2(0)$	13
$g_2(1)$	9
$g_2(3)$	25

The minimum value is 9 and the maximum value is 25.

Scoring:

1 point for finding the critical point
1 point for correct evaluations at 0, 1, 3

1 point for identifying the maximum and minimum

(d) (3 points) Find the minimum and maximum values of $f(x, x)$ for $0 \leq x \leq 3$.

Solution: $g_3(x) = f(x, x) = 5x^2 - 10x + 10$ The derivative is $10x - 10$ so the only critical point is at $x = 1$. Thus:

$g_3(0)$	10
$g_3(1)$	5
$g_3(3)$	25

The minimum value is 5 and the maximum value is 25

Scoring:

1 point for finding the critical point
1 point for correct evaluations at 0, 1, 3

1 point for identifying the maximum and minimum

(e) (2 points) State the maximum and minimum values of f on D.

Solution:

Maximum Value: 5 (1 points)

Minimum Value: 25 (1 points)