Exam 3

Name: ______________________ Section and/or TA: ____________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

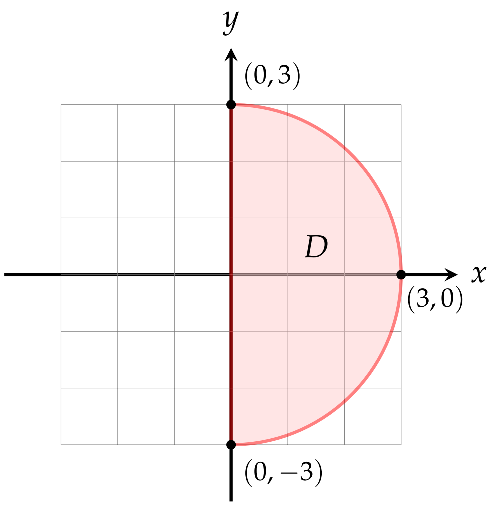
10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. Give the correct expression for $\int\int_D e^{-(x^2+y^2)}\, dA$ if D is right half-disk with center at the origin and radius 3 as shown in the figure.



A. $\int_{-\pi/2}^{\pi/2} \int_0^3 re^{-r^2}\, dr\, d\theta$

B. $\int_{-\pi/2}^{\pi/2} \int_0^3 e^{-r^2}\, dr\, d\theta$

C. $\int_0^{\pi/2} \int_0^3 re^{-r^2}\, dr\, d\theta$

D. $\int_0^{\pi} \int_0^3 e^{-r^2}\, dr\, d\theta$

E. $\int_{-\pi/2}^{\pi/2} \int_{-3}^3 e^{r^2}\, dr\, d\theta$

2. Evaluate the iterated integral $\int_0^1 \int_x^1 \cos(y^2)\, dy\, dx$ by switching the order of integration.

A. $1/3$

B. $\cos(1)$

C. $-1/2$

D. $\sin(1)$

E. $\frac{1}{2}\sin(1)$

3. The iterated integral $\int_0^1 \int_y^1 xy^2 \, dx \, dy$ has the value

   A. $2/5$

   B. $1/10$

   C. $1/15$

   D. $-2/5$

   E. $-1/15$

4. Use cylindrical coordinates to find the volume of the region below the paraboloid $z = 9 - x^2 - y^2$, above the xy-plane and outside the cylinder $x^2 + y^2 = 1$.

   A. 16π

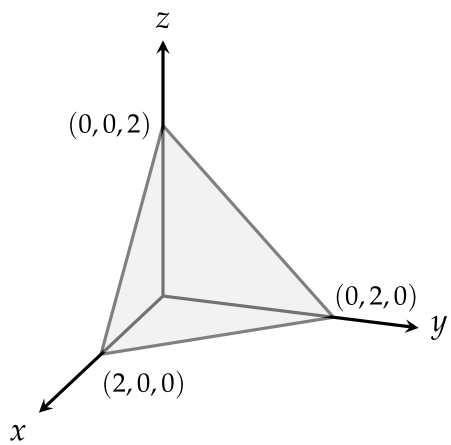
   B. $34\pi/4$

   C. 3π

   D. 32π

   E. 8π

5. Which of the following iterated integrals correctly computes $\int\int\int_R x\, dV$ where R is the tetrahedral region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 2$?



A. $\int_0^2 \int_0^{2-x} \int_0^{x+y+z} x\, dz\, dy\, dx$

B. $\int_0^2 \int_0^{x} \int_0^{x+y+z} x\, dz\, dy\, dx$

C. $\int_0^2 \int_0^{x} \int_0^{2-x-y} x\, dz\, dy\, dx$

D. $\int_0^2 \int_0^{2-x} \int_0^{2-x-y} x\, dz\, dy\, dx$

E. $\int_0^{2-x} \int_0^{2} \int_0^{2-x-y} x\, dz\, dy\, dx$

6. Suppose that E is the region bounded by the spheres

$$x^2 + y^2 + z^2 = 1, \quad x^2 + y^2 + z^2 = 4$$

and having $y \geq 0$. Which of the following expresses the integral

$$\iiint_E x^2y^2\, dV$$

in spherical coordinates?

A. $\int_0^{\pi} \left(\int_0^{\pi} \left(\int_1^2 \rho^6 \sin^5\phi \sin^2\theta \cos^2\theta\, d\rho \right) d\theta \right) d\phi$

B. $\int_0^{\pi/2} \left(\int_0^{\pi} \left(\int_1^4 \rho^6 \sin^5\phi \sin^2\theta \cos^2\theta\, d\rho \right) d\theta \right) d\phi$

C. $\int_0^{\pi} \left(\int_0^{\pi} \left(\int_1^2 \rho^4 \sin^4\phi \sin^2\theta \cos^2\theta\, d\rho \right) d\theta \right) d\phi$

D. $\int_0^{\pi} \left(\int_0^{2\pi} \left(\int_1^2 \rho^6 \sin^4\phi \sin^2\theta \cos^2\theta\, d\rho \right) d\theta \right) d\phi$

E. $\int_0^{\pi} \left(\int_0^{2\pi} \left(\int_1^2 \rho^4 \sin^4\phi \sin^2\theta \cos^2\theta\, d\rho \right) d\theta \right) d\phi$

7. Find the Jacobian determinant for the transformation $T(u,v) = (u^2 - v^2, 2uv)$.
   - A. $2u^3v - 2uv^3$
   - **B.** $4(u^2 + v^2)$
   - C. $u^2 + v^2$
   - D. $2u + 2v$
   - E. $2u - 2v$

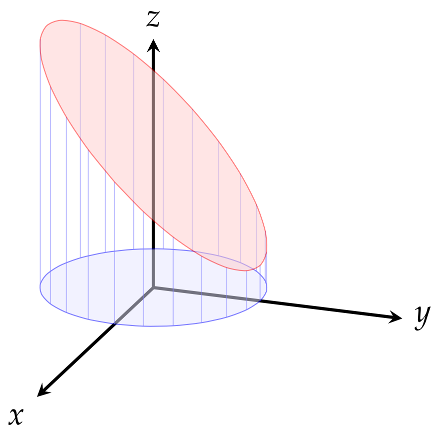
8. Find the (x, y, z) coordinates of the point P with spherical coordinates $\rho = 4$, $\theta = \pi/3$, $\phi = \pi/4$.
   - A. $(\sqrt{6}, \sqrt{2}, 2\sqrt{2})$
   - B. $(\sqrt{6}, \sqrt{6}, 2\sqrt{3})$
   - C. $(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$
   - D. $(\sqrt{6}, 2\sqrt{3}, \sqrt{6})$
   - **E.** $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$

9. Find the gradient vector field of the function $f(x, y, z) = xye^{xyz}$
   - A. $\langle (y + xyz^2)e^{xyz}, (y + x^2yze^{xyz}, x^2y^2e^{xyz} \rangle$
   - B. $\langle (x + xyz^2)e^{xyz}, (x + xy^2z)e^{xyz}, x^2y^2e^{xyz} \rangle$
   - **C.** $\langle (y + xy^2z)e^{xyz}, (x + x^2yz)e^{xyz}, x^2y^2e^{xyz} \rangle$
   - D. $\langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$
   - E. $\langle xe^{xyz}, ye^{xyz}, ze^{xyz} \rangle$

10. Find $\int_C xy\, ds$ if C is the curve $x(t) = \cos t$, $y(t) = \sin t$, and $0 \le t \le \pi/2$.
    - A. 1
    - **B.** $1/2$
    - C. $\pi/2$
    - D. π
    - E. 2π

Free Response Questions

11. (10 points) In this problem we will evaluate the volume of the region S inside the cylinder $x^2 + y^2 = 9$, above the plane $z = 0$ and below the plane $z = 4 - y$.



(a) (4 points) Set up the triple integral to determine this volume via cylindrical coordinates.

Solution:

$$\int\int\int_S 1\, dV = \int_0^{2\pi}\int_0^3\int_0^{4-r\sin\theta} 1\, r\, dz\, dr\, d\theta$$

Scoring:

1 point each for correct limits

1 point for factor r

(b) (6 points) Evaluate the triple integral you found in part a.

Solution:

$$\begin{aligned}\int\int\int_S 1\, dV &= \int_0^{2\pi}\int_0^3 (4 - r\sin\theta)\, r\, dr\, d\theta \\ &= \int_0^{2\pi} (2r^2 - \frac{r^3}{3}\sin\theta\Big|_0^3\, d\theta \\ &= \int_0^{2\pi} 18 - 9\sin\theta\, d\theta \\ &= 18\theta + 9\cos\theta|_0^{2\pi} \\ &= 36\pi\end{aligned}$$

Scoring:

1 point for z-integral

2 point for r integral
2 point for θ integral
1 points for answer

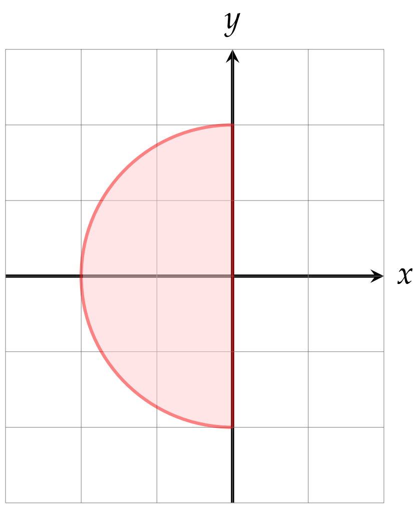
12. (15 points) In this problem, we will evaluate the double integral

$$\int\int_R \sqrt{x^2+y^2}\,dA = \int_{-2}^{0}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx$$

using polar coordinates.

(a) (3 points) Sketch the region R of integration.

Solution: The region is the disk of radius 2 centered at the origin located in quadrant II and III.



Scoring:

1 point for disk of radius 2
1 point for placing half-disk in left half-plane
1 bonus point for quality of sketch

(b) (4 points) Rewrite the region using polar coordinates.
Answer:

_____ $\leq r \leq$ _____

_____ $\leq \theta \leq$ _____

Solution: We have $0 \leq r \leq 2$ and $\pi/2 \leq \theta \leq 3\pi/2$.
medskip
Scoring:

1 point for each of four limits of integration

(c) (4 points) Rewrite the integral using polar coordinates.

Solution: Note $f(x,y) = \sqrt{x^2+y^2}$.
Thus $f(r,\theta) = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r$. So

$$\int_{\pi/2}^{3\pi/2} \int_0^2 r \cdot r \, dr \, d\theta$$

Scoring:

1 point for substitution $x = r\cos\theta$, $y = r\sin\theta$ 1 point for factor of r 2 points for correct limits

(d) (4 points) Evaluate the integral in part (c).

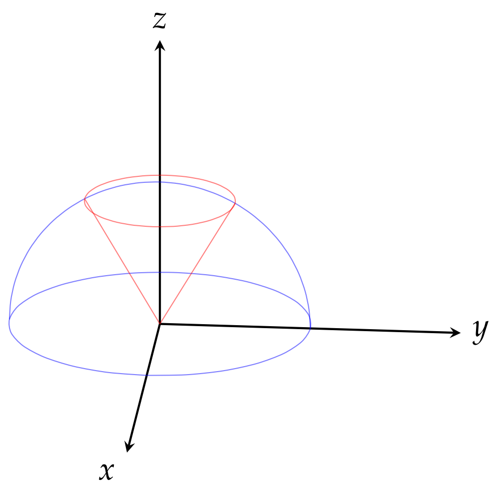
Solution:

$$\int_{\pi/2}^{3\pi/2} \int_0^2 r \cdot r \, dr \, d\theta = \int_{\pi/2}^{3\pi/2} \frac{r^3}{3}\bigg|_0^2 \, d\theta = \int_{\pi/2}^{3\pi/2} \frac{8}{3} \, d\theta$$
$$= \frac{8}{3}\theta\bigg|_{\pi/2}^{3\pi/2} = \frac{8}{3}(3\pi/2 - \pi/2) = \frac{8}{3}\pi.$$

Scoring:

1 point for correct limits
1 point for factor of r
2 points for computation of answer

13. (10 points) The purpose of this problem is to find the volume of the region E bounded below by the cone $z = \sqrt{3(x^2 + y^2)}$ and above by the sphere $x^2 + y^2 + z^2 = 4$.



(a) (3 points) Find the intersection of these two surfaces. Be sure to identify the curve and specify the z coordinate of points on the curve.

Solution: Substituting for z we get $4x^2 + 4y^2 = 4$ or $x^2 + y^2 = 1$. Using the equation of the sphere again we get that $z^2 = 3$ or $z = \sqrt{3}$. Hence, the intersection of the two surfaces is the curve $(x, y, \sqrt{3})$ where $x^2 + y^2 = 1$, i.e., the circle of radius 1 centered at $(0, 0, \sqrt{3})$ and contained in the plane $z = \sqrt{3}$ parallel to the xy plane.

Scoring:

1 point for finding equation of circle
1 point for concluding that $z = \sqrt{3}$
1 point for identifying the circle as contained in the plane $z = \sqrt{3}$

(b) (3 points) Describe the region E in spherical coordinates.

Solution: The volume is a section of the sphere of radius 2 so $0 \leq \rho \leq 2$. The volume has rotational symmetry about the z-axis so $0 \leq \theta \leq 2\pi$. Finally the angle ϕ goes between 0 and $\pi/6$ since the intersection curve has radius 1 and is at height $\sqrt{3}$, so $\tan\phi = \sqrt{3}$.

Scoring:

1 point for each correct range of ρ, θ, and ϕ

Answer:

_____ $\leq \rho \leq$ _____

_____ $\leq \theta \leq$ _____

_____ $\leq \phi \leq$ _____

(c) (4 points) Set up and evaluate the triple integral for the volume of E

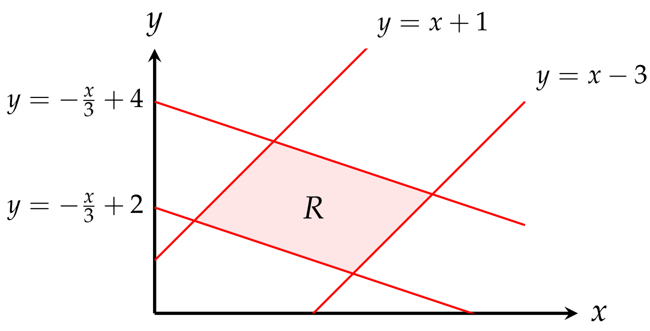
Solution:

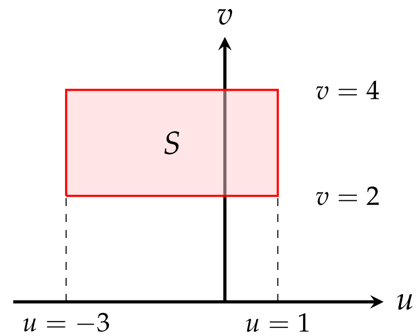
$$\begin{aligned}
\iiint_E 1\,dV &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_0^{\pi/6} \int_0^{2\pi} \frac{8}{3} \sin\phi \, d\theta \, d\phi \\
&= \int_0^{\pi/6} \frac{16\pi}{3} \sin\phi \, d\phi \\
&= \frac{16\pi}{3} \left(\frac{\sqrt{3}}{2} - 1 \right)
\end{aligned}$$

Scoring:

1 point for correct Jacobian factor of $\rho^2 \sin\phi$
1 point for ρ integral
1 point for θ integral
1 point for ϕ integral and final answer

14. The goal of this problem is to compute $\iint_R (x-y)\,dA$ where R is the region bounded by the lines $y = x+1$, $y = x-3$, $y = -x/3+4$, and $y = -x/3+2$ shown below at left.





(a) (4 points) Let

$$u = y - x, \quad v = y + x/3.$$

Show that the lines $y = x+1$ and $y = x-3$ correspond to the lines $u = 1$ and $u = -3$, and the lines $y = -x/3+4$ and $y = -x/3+2$ correspond to $v = 4$ and $v = 2$. Conclude that the region R corresponds to the region S at right.

Solution: The line $y = x - 1$ is $y - x = 1$ or $u = 1$. (1 points)
The line $y = x - 3$ is $y - x = -3$ or $u = -3$. (1 points)
The line $y = -x/3 + 2$ is $v = 2$. (1 points)
The line $y = -x/3 + 4$ is $v = 4$. (1 points)

(b) (4 points) Find formulas for x and y in terms of u and v.

Solution: Solving for x and y gives

$$x = -\frac{3}{4}u + \frac{3}{4}v \qquad \text{(2 points)}$$
$$y = \frac{1}{4}u + \frac{3}{4}v \qquad \text{(2 points)}$$

(c) (2 points) Find the Jacobian determinant J of the map $(u,v) \to (x,y)$.

Solution: The Jacobian determinant is

$$\begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = -\frac{3}{4}$$

Scoring:

1 point for correct set-up of Jacobian
1 point for computation

(d) (5 points) Find $\iint_R (x-y)\,dA$ using the change of variables theorem and integrating over the region S.

Solution: From part (b), $x - y = -u$. Hence

$$\begin{aligned}\iint_R (x-y)\,dA &= \iint_S (-u)\,\frac{3}{4}\,du\,dv \\ &= \int_{-3}^{1}\int_{2}^{4}(-u)\frac{3}{4}\,dv\,du \\ &= \int_{-3}^{1} -\frac{3}{2}u\,du \\ &= \frac{3}{4}(-u^2)\Big|_{-3}^{1} \\ &= 6\end{aligned}$$

Scoring:

1 point for correct substitution $x - y = -u$
1 point for Jacobian factor
1 point for correct limits
2 points for computation and correct answer