Strickland's elementary argument that the ring structure on $\pi_0 L_{KU/2} S^0$ is $\mathbb{Z}_2[x]/(2x, x^2)$.

① Let $KO = KO^\wedge_2$. Start w/ fiber seq $L_{KU/2} S^0 \to KO \xrightarrow{\psi^3 - 1} KO$.

Product w/ fiber seq $* \to KO \to KO$ to get fiber seq

$$L_{KU/2} S^0 \longrightarrow KO \times KO \longrightarrow KO \times KO.$$

Since this is a fiber seq. the following square is a homotopy pullback:

$$\begin{tikzcd} L_{KU/2} S^0 \arrow[r] \arrow[d] & KO \arrow[d, "(1,\psi^3)"] \\ KO \arrow[r, "(1,1)"] & KO \times KO. \end{tikzcd}$$

② Assume
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$
is a htpy pb square of ring spectra, then $\ker(\pi_0 i) \cdot \ker(\pi_0 j) = 0$.

In the square above, $i = j$ and $x \in \ker(\pi_0 i)$.

Why? Extend to the total htpy fiber square

$$\begin{tikzcd} * \arrow[r] \arrow[d] & F(j) \arrow[r] \arrow[d] & F(k) \arrow[d] \\ F(i) \arrow[r] \arrow[d] & A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "k"] \\ F(l) \arrow[r] & C \arrow[r, "l"] & D \end{tikzcd}$$

Let $x \in \ker(\pi_0 i)$ and $y \in \ker(\pi_0 j)$, then xy is in both kernels. Assume $z_i \in F(i)$ and $z_j \in F(j)$ have the property that $z_i \mapsto xy$ and $z_j \mapsto xy$.

Then in the fiber seq $* \to F(i) \vee F(j) \to A$

$(z_i, -z_j) \mapsto 0 \in \pi_0 A$. $\Rightarrow$ $z_i, z_j = 0 \Rightarrow xy = 0$.

③ On the total fiber square.

Since $\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "\ell"] & D \end{tikzcd}$ is a pullback, we can extend to a cube

$$\begin{tikzcd}
F(j) \arrow[rr] \arrow[dd] \arrow[dr] & & F(k) \arrow[dd] \arrow[dr] & \\
& A \arrow[rr] \arrow[dd] & & B \arrow[dd] \\
* \arrow[rr] \arrow[dr] & & * \arrow[dr] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

in which the front, left, and right squares are pullbacks. This implies that the back square is a pullback, which means that $F(j) \to F(k)$ is an equiv. The same argument shows that $F(i) \longrightarrow F(\ell)$ is an equiv.

This argument can also be run in reverse. If $F(j) \longrightarrow F(k)$ is an equivalence, build the cube above. Since pb sqs are pushout sq's, $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$ is a pushout and thus a pb.