

Strickland's elementary argument that the ring structure on $\pi_0 L_{ku/2} S^0$ is $\mathbb{Z}_2[x] / (2x, x^2)$.

① let $KO = KO_2^1$. Start w/ fiber seq $L_{ku/2} S^0 \rightarrow KO \xrightarrow{\psi_1} KO$.

Product w/ fiber seq $* \rightarrow KO \rightarrow KO$ to get fiber seq

$$L_{ku/2} S^0 \rightarrow KO \times KO \rightarrow KO \times KO.$$

Since this is a fiber seq. the following square is a homotopy pullback:

$$\begin{array}{ccc} L_{ku/2} S^0 & \longrightarrow & KO \\ \downarrow & & \downarrow (1, \psi^3) \\ KO & \xrightarrow{(1,1)} & KO \times KO. \end{array}$$

② Assume $A \xrightarrow{i} B$ is a htpy pb square of spectra, $j \downarrow \quad \downarrow k$ then $\ker(\pi_0 i) \cdot \ker(\pi_0 j) = 0$. In the square above, $i=j$ and $x \in \ker(\pi_0 i)$.

Why? Extend to the total htpy fiber square

$$\begin{array}{ccccc} * & \longrightarrow & F(j) & \longrightarrow & F(k) \\ \downarrow & & \downarrow & & \downarrow \\ F(i) & \longrightarrow & A & \xrightarrow{i} & B \\ \downarrow & & \downarrow j & & \downarrow k \\ F(l) & \longrightarrow & C & \xrightarrow{\ell} & D \end{array}$$

let $x \in \ker(\pi_0 i)$ and $y \in \ker(\pi_0 j)$, then xy is in both kernels. Assume $z_i \in F(i)$ and $z_j \in F(j)$ have the property that $z_i \mapsto xy$ and $z_j \mapsto xy$.

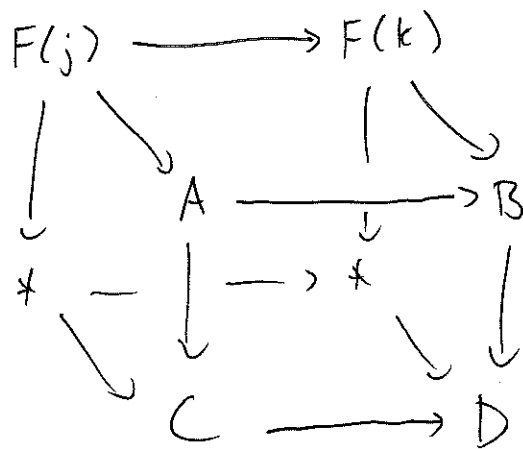
Then in the fiber seq $* \rightarrow F(i) \vee F(j) \rightarrow A$

$$(z_i, -z_j) \mapsto 0 \in \pi_0 A. \Rightarrow z_i, z_j = 0 \Rightarrow xy = 0.$$

③ On the total fiber square.

Since $A \xrightarrow{i} B$

$\begin{array}{ccc} \downarrow j & & \downarrow k \\ C & \xrightarrow{k} & D \end{array}$ is a pullback, we can extend to a cube



in which the front, left, and right squares are pullbacks.

This implies that the back square is a pullback,

which means that $F(j) \rightarrow F(k)$ is an equiv. The same argument shows that $F(i) \rightarrow F(l)$ is an equiv.

This argument can also be run in reverse. If $F(j) \rightarrow F(k)$ is an equivalence, build the cube above. Since pb sqs are pushout sq's, $A \rightarrow B$ is a pushout and thus a pb.

$$\begin{array}{ccc}
 A & \rightarrow & B \\
 \downarrow & & \downarrow \\
 C & \rightarrow & D
 \end{array}$$