Étale htpy thy

Motivating problem: X a scheme. Can we construct a space $|X|$ s.t. $H^*_{ét}(X, A) \cong H^*(|X|, A)$

and $\pi_1^{ét}(X, x) = \pi_1(|X|, |x|)$?

Recall: $X \rightsquigarrow ét(X)$ the étale site: covers are collections of étale morphisms which are jointly surjective.

A is the constant abelian sheaf.

$H^i_{ét}(X, A)$ is the ith right derived functor of Γ.

Ex. $X = \mathrm{spec}(k)$ then $ét(X) = {}^{cts}Gal(\bar{k}/k)$-sets
$\quad \Gamma_k = \lim \Gamma_k/\Gamma_L = \lim Gal(L/k)$

$$H^n_{ét}(X, A) = H^n_{cts}(\Gamma_k, A) = \operatorname{colim} H^n(\Gamma_k/\Gamma_L, A)$$

$$\pi_1^{ét}(X, \mathrm{spec}\,\bar{k}) = \Gamma_k.$$

Cannot seem to solve the problem w/ a space, need a pro-space.

Artin-Mazur: For X geometrically unibranch they construct a pro-homotopy type

$$|X_{ét}| \in \mathrm{Pro}(\mathrm{Ho}(\mathrm{Set}^{\Delta^{op}})) \text{ s.t.}$$

$$H^*_{ét}(X, A) = H^*(|X_{ét}|, A) = \operatorname{colim}_{\alpha} H^*(|X_{ét}|_\alpha, A).$$

$$\pi_1^{ét}(X_{ét}, x) = \pi_1(|X_{ét}|, |x|)$$

and For $X/\mathbb{C}$ nice there is an equiv.

$$\widehat{X(\mathbb{C})} \longrightarrow |X_{ét}|.$$

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Idea: $\mathcal{U} \to X$ a cover

let $N(\mathcal{U})$ be the Čech nerve.

then $\pi_0 N(\mathcal{U}) \in Set^{\Delta^{op}}$

$X \rightsquigarrow$ the "pro"-system $\{\pi_0 N(\mathcal{U})\}_{\text{all covers } \mathcal{U}}$

Problems: 1 Really need hypercovers

2 Not a pro-system (introduce notion of ~~htpy~~ equiv of hypercovers)

Ideally: $|X_{\acute{e}t}| \in Pro(Set^{\Delta^{op}})$ (for htpy fixed pts or a notion of fibration

Friedlander accomplished this by rigidifying the notion of cover.

His construction is limited to the étale site.

General relative solution by Barnea + Schlank.

Let T, S be topoi (ie. cats of sheaves on a site)

A geometric morphism $f: T \to S$ is a pair of adjoint functors $f^* \dashv f_*$

$$f_*: T \to S$$

$$T \leftarrow S : f^*$$

s.t. f^* preserves finite limits.

Ex. A map of spaces $X \xrightarrow{f} Y$ induces

$f_*: Sh(X) \to Sh(Y)$ (direct image)

$f^*: Sh(Y) \to Sh(X)$ (inverse image)

Ex. The final topos is Set.

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There are induced maps

$$f_* : T^{\Delta^{op}} \underset{\perp}{\overset{\longleftarrow}{\longrightarrow}} S^{\Delta^{op}} : f^*$$

and $\mathrm{Pro}(f_*) : \mathrm{Pro}(T^{\Delta^{op}}) \underset{\perp}{\overset{\longleftarrow}{\longrightarrow}} \mathrm{Pro}(S^{\Delta^{op}}) : \mathrm{Pro}(f^*)$

<u>But</u> $\mathrm{Pro}(f^*)$ preserves all limits $\Longrightarrow \exists f_! \dashv \mathrm{Pro}(f^*)$.

Thm: (Barnea-Schlank) For any topos T there is a "projective" model structure on $\mathrm{Pro}(T^{\Delta^{op}})$ s.t. if $T \xrightarrow{f} S$ is a geometric morphism the $f_! \dashv \mathrm{Pro}(f^*)$ is a Quillen pair.

Rk: An injective model struct. was known on $T^{\Delta^{op}}$ by Jardine and Joyal.

Def: Given $T \xrightarrow{f} S$ a geom. mor. of topoi, the realization of T relative to S is

$$|T|_S := \mathbb{L}f_!(*) \in \mathrm{Pro}(S^{\Delta^{op}}).$$

Motivation: let $S = \mathrm{Set}$

$$\begin{aligned} H^n(|T|_{\mathrm{Set}}, A) &= [|T|_{\mathrm{Set}}, K(A,n)] \\ &= [\mathbb{L}f_!(*), K(A,n)] \\ &= [*, \mathbb{R}f^*(K(A,n))] \\ &= [*, K(f^*A, n)] \\ &= [*, \mathcal{U}(f^*A[n])] \\ &= [F(*), f^*A[n]] = H^n(T, f^*A). \end{aligned}$$

Cor: X any scheme, $X_{ét} = Sh(ét(X))$

$$X_{ét} \underset{\Gamma_*}{\overset{\Gamma^*}{\rightleftarrows}} Set \rightsquigarrow |X_{ét}|_{Set} \in Pro(Set^{\Delta^{op}})$$

Gives AM-étale ho. type in $Pro(Ho(Set^{\Delta^{op}}))$.

Examples: 1) $Pre(D) \underset{colim}{\overset{const}{\rightleftarrows}} Set \quad |Pre(D)|_{Set} = N(D)$.

2) $Sh(X) \underset{\Gamma_*}{\overset{\Gamma^*}{\rightleftarrows}} Set \quad |Sh(X)|_{Set} = X$ if X is nice.

3) $X = \operatorname{spec} \mathbb{R} \quad \Gamma_{\mathbb{R}} = \mathbb{Z}/2$

by example 1 $|X_{ét}|_{Set} = B\mathbb{Z}/2$.

4) If $X = \operatorname{Spec} k$ then $X_{ét} \overset{cts}{=} \Gamma_k$-sets

$$|X_{ét}|_{Set} = \{B\Gamma_k/H\}_{H \subseteq \Gamma_k \text{ open}}$$

5)
$$\begin{array}{c} \operatorname{Spec} k \\ \downarrow id \\ \operatorname{Spec} k \end{array} \rightsquigarrow |X_{ét}|_{X_{ét}} \simeq \{E\Gamma_k/H\}_{H \subseteq \Gamma_k \text{ open}}$$

6)
$$\begin{array}{c} X \\ \downarrow \\ \operatorname{Spec} k \end{array} \rightsquigarrow X_{ét} \longrightarrow \Gamma_k\text{-sets}$$

$$|X_{ét}|_{\Gamma_k\text{-set}} \curvearrowleft \Gamma_k.$$

7) Obstructions to rational pts.

$$\begin{array}{c} X \\ \downarrow \uparrow \\ \operatorname{Spec} k \end{array}$$

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