EXCELLENT RINGS IN TRANSCHROMATIC HOMOTOPY THEORY

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Abstract. The purpose of this note is to verify that several basic rings appearing in transchromatic homotopy theory are Noetherian excellent normal domains and thus amenable to standard techniques from commutative algebra. In particular, we show that the coefficients of iterated localizations of Morava E-theory at the Morava K-theories are normal domains and also that the coefficients in the transchromatic character map for a fixed group form a normal domain.

1. Introduction

Excellent rings were introduced by Grothendieck as a well-behaved class of commutative Noetherian rings general enough for the purposes of arithmetic and algebraic geometry, while excluding several pathological examples of Noetherian rings found by Nagata. In particular, the collection of excellent rings is closed under localization and completion.

These algebraic operations describe the effect on coefficient rings of the derived localizations appearing in stable homotopy theory. Most prominently, such Bousfield localizations occur when comparing different chromatic layers of the stable homotopy category, a subject known as transchromatic homotopy theory. The main goal of this note is to demonstrate that important rings appearing in transchromatic homotopy theory are built from excellent rings, and hence surprisingly well-behaved. Specifically, although these rings are rather complicated algebraically and in general not regular, we prove that they are integral domains and thus regular in codimension 1. However, establishing these fundamental properties directly turned out to be considerably more difficult than anticipated, which led us to employ the theory of excellent rings instead. We hope that the methods used here will prove useful in tackling similar problems in related contexts.

Our first result concerns the rings $L_{t,n} = \pi_0 L_{K(t)} E_n$ obtained as the localization of Morava E-theory E_n of height n at the Morava K-theory $K(t)$ of height $t < n$. This is a fundamental example of a transchromatic ring, since the localization map

$$E_n \longrightarrow L_{K(t)} E_n$$

shifts chromatic height from n to t. These rings were studied in detail by Mazel–Gee, Peterson, and Stapleton in [MGPS15] using the theory of pipe rings. In particular, Theorem 36 of [MGPS15] shows that $\pi_0 E_n \to L_{t,n}$ represents a natural moduli problem. Technicalities aside, the moduli problem associates to a pair of (properly topologized) rings $R \to S$ the groupoid of deformations of a fixed height n formal group over a perfect field k to R with the property that the pullback to S has height t up to a notion of $\star$-isomorphism.

We complement their work by showing that the $L_{t,n}$ are well-behaved from the point of view of commutative algebra.

Theorem (Proposition 3.2)**.** *The ring $L_{t,n}$ is a Noetherian excellent normal domain. More generally, the same conclusion holds for any iteration of localizations of E_n at Morava K-theories.*

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As a consequence, $L_{t,n}$ has the cancellation property, which is crucial in applications as for example in [BS15]. Moreover, this result about $L_{t,n}$ forms the basis for deducing similar properties of other prominent rings appearing in transchromatic homotopy theory. Specifically, we study the completion at I_t of the transchromatic character rings $C_{t,k}$

$$\hat{C}_{t,k} = (C_{t,k})^{\wedge}_{I_t}.$$

that were first introduced in [Sta13]. For a fixed finite group G there exists a $k \geq 0$ so that $\hat{C}_{t,k}$ may be used to build a transchromatic character map

$$\hat{C}_{t,k} \otimes_{E_n^0} E_n^0(BG) \xrightarrow{\cong} \hat{C}_{t,k} \otimes_{L_{t,n}} L_{K(t)}E_n^0(BG^{B\mathbb{Z}_p^{n-t}}).$$

Theorem (Corollary 3.8 and Corollary 3.9)**.** *The transchromatic character ring $\hat{C}_{t,k}$ is a Noetherian excellent normal domain for all t and k. The colimit* $\operatorname{colim}_k \hat{C}_{t,k}$ *is normal.*

There are two key ingredients in the proof of this theorem: Firstly, a recent theorem of Gabber–Kurano–Shimomoto on the ideal-adic completion of excellent rings. Secondly, we use an identification of $\hat{C}_{t,k}$ with a localization and completion of a certain ring of Drinfeld level structure on the formal group associated to E_n, thereby providing a new perspective on these transchromatic character rings.

Conventions and references. We rely heavily on a number of results from commutative algebra that cannot be found in standard textbooks. Rather than locating the earliest published reference for each of the facts used here, we will always refer to the stacks project [Sta15]. All rings in this note are assumed to be commutative and all ideals are taken to be finitely generated.

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2. Excellent rings

We start this section with an example illustrating that localizations of complete regular local Noetherian rings are more complicated than what one might expect. This shows that, while the localization or completion of a regular local ring is again regular, the class of regular local rings is not closed under these operations.

Example 2.1. Let k be a field and consider $R = k[\![x, y]\!]$. The ring $A = y^{-1}R \cong k[\![x]\!]((y))$ is not local and, in particular, it is not isomorphic to the regular local ring $k((y))[\![x]\!]$. Indeed, we claim that both (x) and $(x - y)$ are different maximal ideals in A. The ideal (x) is clearly maximal, so it remains to show that $(x - y)$ is maximal as well. To this end, note that

$$A/(x - y) \cong k((x)),$$

which is a field, hence $(x - y) \subset A$ is maximal as claimed. In contrast, $(x - y) = x(1 - y/x)$ has formal inverse $1/x \cdot \sum_{i=0}^{\infty}(y/x)^i$, so $x - y$ is a unit in $k((y))[\![x]\!]$.

After completion, this subtlety disappears: in fact, $A^{\wedge}_{(x)} \cong k((y))[\![x]\!]$, see [MGPS15, B.2].

The Auslander–Buchsbaum theorem asserts that regular local rings are unique factorization domains, but in light of the previous example, we may not expect this property to be preserved under the operations appearing in transchromatic theory. Instead, we will study the larger class of normal rings, which corresponds to regularity in codimension 1 via Serre's criterion.

To this end, recall that a normal domain is a domain which is integrally closed in its quotient field. A ring R is called normal if the localizations $R_{\mathfrak{p}}$ are normal domains for all primes ideals $\mathfrak{p} \subset R$. The following lemma collects the key properties of normal rings that we will apply in this note.

Lemma 2.2. *Let R be a commutative ring.*

(1) If R is regular, then R is normal.
(2) If R is normal, then any of its localizations is normal.
(3) Filtered colimits of normal rings are normal.
(4) If R is Noetherian and normal, then it is a finite product of normal domains.

Proof. The first claim is [Sta15, Tag 0567], the second one is [Sta15, Tag 037C], and the third one is [Sta15, Tag 037D]. To prove the last claim we apply [Sta15, Tag 030C], so that we have to check that R is reduced and contains only finitely many minimal primes. Because R is Noetherian, the second condition is satisfied. For the first one, note that, since R is normal it is reduced, as being reduced is a local property and R is locally a domain. □

The definition of an excellent ring is rather involved. For the convenience of the reader, we give this definition; we will assume that the reader is familiar with the definition of a regular local ring.

Definition 2.3. Recall the following definitions for a commutative ring R:

(1) A ring R is a G-ring if for all prime ideals $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is a local G-ring. A local ring $(R, \mathfrak{m})$ is a local G-ring if the completion map $R \to R^{\wedge}_{\mathfrak{m}}$ is a regular morphism. This means that the map is flat and for all primes $\mathfrak{p} \subset R$, $\kappa(\mathfrak{p}) \otimes_R R^{\wedge}_{\mathfrak{m}}$ is Noetherian and geometrically regular over $\kappa(\mathfrak{p})$. A k-algebra R is geometrically regular over k if for every finite extension $k \subset K$, $R \otimes_k K$ is regular, where a ring is regular if it is locally regular.
(2) A ring R is J-2 if for all finite type extensions $R \to S$ the ring S is J-1. A ring R is J-1 if the subset of points of $\mathfrak{p} \in \operatorname{Spec}(R)$ with the property that $R_{\mathfrak{p}}$ is regular local is open.
(3) A ring R is universally catenary if it is Noetherian and for every finite type extension $R \to S$, the ring S is catenary. A ring R is catenary if for all pairs of primes ideals $\mathfrak{q} \subset \mathfrak{p} \subset R$, all maximal chains of prime ideals $\mathfrak{q} = P_0 \subset P_1 \subset \ldots \subset P_l = \mathfrak{p}$ have the same length.

Finally, a ring R is called excellent if it is Noetherian, a G-ring, J-2, and universally catenary.

For example, fields, Dedekind domains with characteristic 0 quotient field, and all complete local Noetherian rings are excellent, see [Sta15, Tag 07QW]. Moreover, the same reference shows that any algebra of finite type over an excellent ring is excellent.

The next proposition essentially generalizes [Sta15, Tag 0C23] to non-local rings. The purpose is to show that the collection of Noetherian excellent normal rings is closed under the operations of localization at a multiplicatively closed set and completion at a prime ideal. The key point is that we do not assume that our rings are local, as the rings that naturally arise in transchromatic homotopy theory are often not local.

Proposition 2.4. *Suppose R is a Noetherian excellent normal ring, $\mathfrak{p} \subset R$ is a prime ideal, and $S \subseteq R \setminus \mathfrak{p}$ is multiplicatively closed, then $A = (R[S^{-1}])^{\wedge}_{\mathfrak{p}}$ is a Noetherian excellent normal domain.*

Proof. By [Sta15, Tag 07QU] and Part (2) of Lemma 2.2, $R[S^{-1}]$ is Noetherian, excellent, and normal and $\mathfrak{p}R[S^{-1}]$ is prime, thus we may assume without loss of generality that $S = \{1\}$. Consider the canonical map $f\colon R \to A = R^{\wedge}_{\mathfrak{p}}$. Since R is excellent and thus a G-ring, the map f is regular by [Sta15, Tag 0AH2]. The completion of a Noetherian ring is Noetherian, so we may apply [Sta15, Tag 0C22] and Part (1) of Lemma 2.2 to deduce that A is normal as desired. Since A is Noetherian and normal, Part (4) of Lemma 2.2 implies that it is a product of finitely many domains $A_1, \ldots, A_l$. Also, $\mathfrak{p}A$ is prime. Now we have canonical isomorphisms

$$A \cong A_{\mathfrak{p}}^{\wedge} \cong \prod_i (A_i)_{\mathfrak{p}}^{\wedge}.$$

By the structure theory of prime ideals in products, only one of the factors will survive, hence $A \cong A_i$ for some i. Finally, the completion of an excellent ring is excellent by a recent theorem of Gabber–Kurano–Shimomoto [KS16]. □

Remark 2.5. In virtue of [Sta15, Tag 07PW], the proof of the above proposition does not obviously work if we assume that R is a Noetherian normal G-ring. We see that the conditions on R all conspire to make the proof go through.

Proposition 2.6. *Let R be a domain, let $\mathfrak{p} = (r_1, \ldots, r_l) \subset R$ be a prime ideal generated by a regular sequence, and let $\mathfrak{q} = (r_1, \ldots, r_m)$ for $m < l$, then $\mathfrak{q}$ is a prime ideal.*

Proof. Since R is a domain, the localization map $R \to R_{\mathfrak{p}}$ is injective. The ring $R_{\mathfrak{p}}$ is regular local with system of parameters given by $r_1, \ldots, r_l$. In such a situation $R_{\mathfrak{p}}/(r_1, \ldots, r_m)$ is prime for any $1 \leq m \leq l$. Now the preimage of $(r_1, \ldots, r_m) \subset R_{\mathfrak{p}}$ is the ideal generated by $(r_1, \ldots, r_m) \subset R$ since the localization map $R \to R_{\mathfrak{p}}$ is injective. □

3. Rings in transchromatic homotopy theory

Throughout this section, we fix a prime p and height $n \geq 0$. Recall that Morava E-theory E_n is an even periodic $\mathbb{E}_\infty$-ring spectrum with coefficients

$$E_n^* := \pi_{-*}E_n \cong \mathbb{W}k[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}],$$

where $\mathbb{W}k$ is the ring of Witt vectors on a perfect field k of characteristic p, the u_i's are in degree 0 and u has degree 2. Note that $\pi_0 E_n = E_n^0$ is a complete regular local Noetherian ring, so in particular an excellent domain. Furthermore, let $K(n)$ be Morava K-theory of height n with coefficients $K(n)_* \cong k[u^{\pm 1}]$ and denote by $L_{K(n)}$ the corresponding Bousfield localization functor. If $m < n$, then $L_{K(n)}L_{K(m)} = 0$, but the composite $L_{K(m)}L_{K(n)}$ is non-trivial and encodes much of the structure of transchromatic homotopy theory, see for example [Hov95].

Since all spectra involved are even periodic, we will restrict attention to the degree 0 part of the homotopy groups. In particular, an even periodic module M over an even periodic $\mathbb{E}_1$-ring spectrum A is said to be flat if $\pi_0 M$ is flat as $\pi_0 A$-module. This definition is compatible with the one given in [BF15].

Lemma 3.1. *Given a sequence $0 \leq t_1 \leq \cdots \leq t_i \leq n$ of integers, the canonical localization map*

$$M \longrightarrow L_{K(t_1)} \cdots L_{K(t_i)} M$$

is flat for any flat E_n-module M.

Proof. We will prove this by induction on the number i of integers in the sequence. If $i = 0$, the claim is trivial, so suppose it is proven for all sequences of numbers $t_2 \leq \cdots \leq t_i$ and let $t_1 \leq t_2$; for simplicity, write $N = L_{K(t_2)} \cdots L_{K(t_i)} M$. By assumption, $\pi_0 N$ is a flat E_n^0-module, so [BS16, Cor. 3.10] shows that

$$\pi_0 L_{K(t_1)} N \cong (\pi_0 N[u_{t_1}^{-1}])_{I_{t_1}}^{\wedge},$$

where I_t denotes the ideal $(p, u_1, \ldots, u_{t-1})$. Localization is exact, hence $\pi_0 N[u_{t_1}^{-1}]$ is flat over E_n^0. Therefore, an unpublished theorem of Hovey, proven in [BF15, Prop. A.15], implies that $\pi_0 L_{K(t_1)} N$ is flat over E_n^0 as claimed. □

To simplify notation, we shall write $L_{K(T)}$ for the composite functor $L_{K(t_1)} \cdots L_{K(t_i)}$ for any sequence $T = (t_1, \ldots, t_i)$ of integers. Note that, if there is j with $t_{j-1} > t_j$ in T, then $L_{K(T)} \simeq 0$.

Proposition 3.2. *The ring $\pi_0 L_{K(T)}E_n$ is a Noetherian excellent normal domain for all finite non-increasing sequences T of non-negative integers.*

Proof. We proceed by induction, the base case being clear. Suppose that the claim has been proven for all sequences of length at most $i-1$ and consider a sequence $T = \{t_1 \cup T'\}$ with $T' = (t_2, \ldots, t_i)$ of length $i-1$. Write $R = L_{K(T')}E_n$, so $\pi_0 R$ is a Noetherian excellent normal domain by hypothesis. As is shown in the proof of Lemma 3.1, there is an isomorphism $\pi_0 L_{K(t_1)}R \cong (\pi_0 R[u_{t_1}^{-1}])^{\wedge}_{I_{t_1}}$. The ideal $I_{t_1} \subset I_{t_2}$ satisfies the condition of Proposition 2.6 and therefore is prime. Thus the triple $(\pi_0 R, \{u_{t_1}, u_{t_1}^2, \ldots\}, I_{t_1})$ satisfies the assumptions of Proposition 2.4, hence $\pi_0 L_{K(t_1)}R$ is a Noetherian excellent normal domain. □

As a special case of Proposition 3.2, we immediately obtain the following corollary, which was used in the proof of [BS15, Lem. 4.4] and provided the original motivation for this note:

Corollary 3.3. *The ring $L_t = \pi_0 L_{K(t)}E_n$ is an excellent domain for all t and n. In particular, L_t has the cancellation property.*

In [BS15], we also studied a variant $F_t = L_{K(t)}((E_n)_{I_t})$ of L_t, where the localization $(-)_{I_t}$ is understood in the ring-theoretic sense, i.e., as inverting the complement of I_t.

Corollary 3.4. *The ring $\pi_0 F_t$ is an excellent domain.*

Proof. We see as in the proof of Proposition 3.2 that the (degree 0) coefficients of F_t are given by $((E_n^0)_{I_t})^{\wedge}_{I_t}$, to which we can apply Proposition 2.4. □

We now turn to the rings that feature prominently in the transchromatic character theory of Hopkins, Kuhn, and Ravenel [HKR00], as well as its generalizations [Sta13] and [BS16]. To this end, we quickly review the definition and role of the coefficient ring for transchromatic character theory. For any integer $0 \le t \le n$, let $\mathbb{G}_{L_{K(t)}E_n}$ be the formal group associated to the natural map $MU \to E_n \to L_{K(t)}E_n$, viewed as a p-divisible group. In [Sta13], an L_t-algebra called C_t is defined which carries the universal isomorphism of p-divisible groups

$$C_t \otimes \mathbb{G}_{E_n} \cong (C_t \otimes \mathbb{G}_{L_{K(t)}E_n}) \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{n-t}.$$

Note that both L_t and C_t also depend on n. From the perspective of stable homotopy theory, C_t is useful because there is a canonical isomorphism (the character map)

$$C_t \otimes_{E_n^0} E_n^0(BG) \xrightarrow{\cong} C_t \otimes_{\pi_0 L_{K(t)}E_n} L_{K(t)}E_n^0(\mathcal{L}^{n-t}BG),$$

where $\mathcal{L}$ denotes the (p-adic) free loop space. The ring C_t is a colimit of smaller rings $C_t = \operatorname{colim}_k C_{t,k}$. With p and n fixed implicitly, denote the finite abelian group $(\mathbb{Z}/p^k)^{n-t}$ by $\Lambda_{k,t}$. The ring $C_{t,k}$ is a localization of $L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)$. The ring $E_n^0(B\Lambda_{k,t}^*)$ carries the universal homomorphism

$$\Lambda_{k,t} \to \mathbb{G}_{E_n}.$$

Recall from [Sta13, Section 2] that the p-divisible group $L_t \otimes_{E_n^0} \mathbb{G}_{E_n}$ is the middle term in a short exact sequence

$$0 \to \mathbb{G}_{L_t} \to L_t \otimes_{E_n^0} \mathbb{G}_{E_n} \to \mathbb{G}_{et} \to 0,$$

where $\mathbb{G}_{et}$ is a height $n-t$ étale p-divisible group.

Let $T_{t,k} \subset E_n^0(B\Lambda_{k,t}^*)$ be the multiplicative subset generated by the nonzero image of the canonical map

$$\Lambda_{k,t} \to \mathbb{G}_{E_n}(E_n^0(B\Lambda_{k,t}^*)).$$

The nonzero image of this map has an explicit description in terms of a coordinate, as we shall explain now. After fixing an isomorphism $\mathcal{O}_{\mathbb{G}_{E_n}} \cong E_n^0[\![x]\!]$, there is an induced isomorphism

$$E_n^0(B\Lambda_{k,t}^*) \cong E_n^0[\![x_1, \ldots, x_t]\!]/([p^k](x_1), \ldots, [p^k](x_t)),$$

where $[p^k](x)$ is the p^k-series of the formal group law determined by the coordinate. The nonzero image of $\Lambda_{k,t}$ can be described as the set of nonzero sums

$$[i_1](x_1) +_{\mathbb{G}_{E_n}} \ldots +_{\mathbb{G}_{E_n}} [i_t](x_t).$$

Of course, we may view $T_{k,t}$ as a subset of $L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)$.

Let $S_{t,k} \subset L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)$ be the multiplicative subset generated by the nonzero image of the canonical map

$$\Lambda_{k,t} \to (L_t \otimes_{E_n^0} \mathbb{G}_{E_n})(L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)) \to \mathbb{G}_{et}(L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)).$$

The ring $C_{t,k}$ is defined to be $S_{t,k}^{-1}(L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*))$. Instead of working with this ring, we will work with the mild variation obtained by completing at I_t

$$\hat{C}_{t,k} = (C_{t,k})^{\wedge}_{I_t} = (S_{t,k}^{-1}(L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)))^{\wedge}_{I_t}.$$

This ring is an L_t-algebra in a canonical way and it corepresents a certain functor on the category of continuous (with respect to I_t) L_t-algebras. The functor associates to a continuous L_t-algebra R the set of isomorphisms under $R \otimes \mathbb{G}_{L_t}[p^k]$ of the form

$$R \otimes \mathbb{G}_{L_t}[p^k] \oplus \Lambda_{k,t} \xrightarrow{\cong} R \otimes \mathbb{G}_{E_n}[p^k].$$

This follows immediately from Proposition 2.17 in [Sta13].

For a finite abelian group A, the scheme of A-level structures in the formal group $\mathbb{G}_{E_n}$ of Morava E-theory is represented by a ring D_A:

$$\mathrm{Level}(A, \mathbb{G}_{E_n}) \cong \mathrm{Spf}(D_A).$$

This ring was introduced by Drinfeld [Dri74]. It was also studied further by Strickland in [Str97] and first applied to the study of Morava E-theory in [And95].

Lemma 3.5. *The ring $D_{\Lambda_{k,t}}$ is a Noetherian excellent normal domain.*

Proof. The ring $D_{\Lambda_{k,t}}$ is a module-finite extension of E_n^0. This immediately implies that it is Noetherian and excellent. Drinfeld proves that it is regular local and this implies that it is a normal domain. □

Let A^* be the Pontryagin dual of A. The rings $E_n^0(B\Lambda_{k,t}^*)$ and $D_{\Lambda_{k,t}}$ are closely related. There is a canonical surjective map

$$\pi\colon E_n^0(B\Lambda_{k,t}^*) \twoheadrightarrow D_{\Lambda_{k,t}}$$

and the kernel is understood by the proof of Proposition 4.3 in [Dri74]. It is generated by power series $f_i(x_1, \ldots, x_i)$ for $1 \leq i \leq n-t$, where

$$f_i(x_1, \ldots, x_i) = \frac{[p^k](x_i)}{g_i(x_1, \ldots, x_i)}$$

and

$$g_i(x_1, \ldots, x_i) = \prod_{(j_1, \ldots, j_{i-1}) \in \Lambda_{k,i-1}} \big(x_i - ([j_1](x_1) +_{\mathbb{G}_{E_n}} \ldots +_{\mathbb{G}_{E_n}} [j_{i-1}](x_{i-1}))\big).$$

Proposition 3.6. *There is a canonical isomorphism*

$$\hat{C}_{t,k} \cong (T_{t,k}^{-1}(L_t \otimes_{E_n^0} D_{\Lambda_{k,t}}))^{\wedge}_{I_t}.$$

Proof. This will be proved in two steps. First, inverting $T_{t,k}$ and inverting $S_{t,k}$ in $L_t \otimes_{E_n^0} E_n^0(B\Lambda_{k,t}^*)$ give the same ring after I_t-completion. Secondly, we show that inverting $T_{t,k}$ in $E_n^0(B\Lambda_{k,t}^*)$ kills the kernel of π.

It suffices to prove the first claim after taking the quotient by I_t. By [Sta13, proof of Proposition 2.5], after taking the quotient, the ring of functions applied to the quotient

$$L_t \otimes_{E_n^0} \mathbb{G}_{E_n}[p^k] \to \mathbb{G}_{et}[p^k]$$

sends the coordinate y of $\mathbb{G}_{et}[p^k]$ to the function $x^{p^{kt}}$ on $L_t \otimes_{E_n^0} \mathbb{G}_{E_n}[p^k]$. Thus the set $T_{t,k}$ is the p^{kt} powers of the elements in $S_{t,k}$. Inverting an element is equivalent to inverting any of its powers.

Since $a -_{\mathbb{G}_{E_n}} b$ is a unit multiple of $a - b$ (for any elements a, b in the maximal ideal of a complete local ring), it follows from Drinfeld's description of the kernel of π that inverting $T_{t,k}$ in $E_n^0(B\Lambda_{k,t}^*)$ kills the kernel of π. Since $D_{\Lambda_{k,t}}$ is a quotient of $E_n^0(B\Lambda_{k,t}^*)$, there is an isomorphism

$$T_{t,k}^{-1} E_n^0(B\Lambda_{k,t}^*) \cong T_{t,k}^{-1} D_{\Lambda_{k,t}}.$$

□

Remark 3.7. Theorem 36 of [MGPS15] gives a moduli interpretation of the map $\pi_0 E_n \to L_t$. It would be satisfying to give a moduli interpretation of $D_{\Lambda_{k,t}} \to \hat{C}_t$ in terms of pipe rings. The Drinfeld ring $D_{\Lambda_{k,t}}$ has a nice interpretation in terms of deformations equipped with $\Lambda_{k,t}$-level structures up to compatible $\star$-isomorphisms. A moduli interpretation of the map $D_{\Lambda_{k,t}} \to \hat{C}_t$ would likely associate to a (properly topologized) pair $R \to S$ the groupoid with objects deformations of our fixed height n formal group over k to R equipped with $\Lambda_{k,t}$-level structures and such that the pullback as p-divisible groups to S induces an isomorphism of $\Lambda_{k,t}$ with the étale $[p^k]$ torsion of the étale part. In order to achieve this, one would need to develop the theory of p-divisible groups over pipe rings.

Corollary 3.8. *For all k, the rings $\hat{C}_{t,k}$ are Noetherian excellent normal domains.*

Proof. Since $D_{\Lambda_{k,t}}$ is a Noetherian excellent normal domain and $\hat{C}_{t,k}$ can be constructed from $D_{\Lambda_{k,t}}$ by iterated localization and completion, Proposition 2.4 applies. □

Corollary 3.9. *The ring $\operatorname{colim}_k \hat{C}_{t,k}$, which receives a canonical map from C_t, is normal.*

Proof. This is an immediate consequence of Corollary 3.8 and Part (3) of Lemma 2.2. □

For more many purposes in transchromatic homotopy theory (e.g., [BS16]), it is more convenient to work with the completion of $\operatorname{colim}_k \hat{C}_{t,k}$, so we end this note with the following question.

Question 3.10. *What can be said about $(\operatorname{colim}_k \hat{C}_{t,k})^{\wedge}_{I_t}$?*

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