

TRANSFER IDEALS AND TORSION IN THE MORAVA E -THEORY OF ABELIAN GROUPS

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ABSTRACT. Let A be a finite abelian p group of rank at least 2. We show that $E^0(BA)/I_{tr}$, the quotient of the Morava E -cohomology of A by the ideal generated by the image of the transfers along all proper subgroups, contains p -torsion. The proof makes use of transchromatic character theory.

1. INTRODUCTION

The close relationship between the Morava E -cohomology of spaces and algebraic geometry has played an important role in problems related to power operations for Morava E -theory, character theory for Morava E -theory, and understanding H_∞ -ring maps between H_∞ -ring spectra and E . Let \mathbb{G} be the universal deformation formal group associated to E and write I_{tr} for the transfer ideal. A fundamental result of Strickland's in [Str98] shows that $E^0(B\Sigma_{p^k})/I_{tr}$ corepresents the formal scheme parametrizing subgroup schemes of order p^k in \mathbb{G} . For instance, this theorem plays a vital role in Rezk's study [Rez09, Rez12] of the Dyer–Lashof algebra for Morava E -theory. One of the key ingredients in Strickland's proof is the fact that $E^0(B\Sigma_{p^k})/I_{tr}$ is a free E^0 -module.

Let $\text{Level}(A^*, \mathbb{G})$ be the formal scheme of A^* -level structures in \mathbb{G} , where A^* is the Pontryagin dual of a finite abelian p -group A . In [Dri74], it is shown that the ring of functions on $\text{Level}(A^*, \mathbb{G})$ is a finitely generated free E^0 -module. Proposition 7.5 in [AHS04] implies that there is a close relationship between $E^0(BA)/I_{tr}$ and the ring of functions on $\text{Level}(A^*, \mathbb{G})$. In fact, if we let $(E^0(BA)/I_{tr})^{\text{free}}$ be the image of $E^0(BA)/I_{tr}$ in $\mathbb{Q} \otimes E^0(BA)/I_{tr}$, then there is a canonical isomorphism

$$\text{Spf}((E^0(BA)/I_{tr})^{\text{free}}) \cong \text{Level}(A^*, \mathbb{G}).$$

For a transitive abelian subgroup $A \subseteq \Sigma_{p^k}$, Ando, Hopkins, and Strickland use this isomorphism to give an algebro-geometric description of the power operation

$$E^0 \xrightarrow{P_{p^k}/I_{tr}} E^0(B\Sigma_{p^k})/I_{tr} \rightarrow E^0(BA)/I_{tr} \rightarrow (E^0(BA)/I_{tr})^{\text{free}}.$$

This motivates the question of whether $E^0(BA)/I_{tr}$ contains p -torsion, i.e., if

$$(E^0(BA)/I_{tr})^{\text{free}} \not\cong E^0(BA)/I_{tr}.$$

These two rings are isomorphic when A is cyclic. In contrast, the main result of this note implies that they are distinct when the rank of A is greater than 1.

Theorem (Theorem 3.1). *Let A be a finite abelian p -group with $\text{rank}(A) \geq 2$. Let $I_{tr} \subset E^0(BA)$ be the ideal generated by the image of transfers along all proper subgroups of A . The E^0 -algebra $E^0(BA)/I_{tr}$ contains p -torsion.*

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In particular, this answers a question raised in [HS20, Remark 5.3]. Our proof crucially uses the transchromatic character maps of [BS16] to first reduce to a question about p -adic K -theory. We then use classical character theory to show that $K_p^0(BA)/I_{tr}$ is non-zero and contains torsion when $\text{rank}(A) \geq 2$.

2. p -ADIC K -THEORY OF FINITE ABELIAN p -GROUPS AND TORSION

Fix a prime p and let K_p denote p -adic K -theory. Given a finite p -group G , there is a canonical isomorphism

$$\mathbb{Z}_p \otimes R(G) \xrightarrow{\cong} K_p^0(BG),$$

where $R(G)$ is the representation ring of G . It follows that there is an isomorphism

$$K_p^0(B\mathbb{Z}/p^k) \cong \mathbb{Z}_p[x]/(x^{p^k} - 1) \cong \mathbb{Z}_p[x]/([p^k](x)),$$

where $[p^k](x)$ is the p^k -series for the multiplicative formal group law. Since $R(G \times H) \cong R(G) \otimes R(H)$ for finite groups G and H , the isomorphisms above give an explicit description of $K_p^0(BA)$ for any finite abelian p -groups A .

Character theory for p -adic K -theory was developed by Adams in [Ada78, Section 2] and generalized to all Morava E -theories by Hopkins, Kuhn, and Ravenel in [HKR00]. Let $D = \text{colim}_{i \geq 0} \mathbb{Z}_p(\zeta_{p^i})$, the ring obtained from \mathbb{Z}_p by adjoining all of the p^i th roots of unity as i varies. Let $C_0 = \mathbb{Q} \otimes D$. For a finite group G , the character map is a canonical inclusion of \mathbb{Z}_p -algebras

$$K_p^0(BG) \hookrightarrow Cl_p(G, C_0),$$

where $Cl_p(G, C_0)$ is the ring of C_0 -valued functions on the set of conjugacy classes of p -power order elements in G . By construction, the character map factors through $Cl_p(G, D)$, the ring of D -valued functions on the set of conjugacy classes of p -power order elements in G , giving

$$K_p^0(BG) \hookrightarrow Cl_p(G, D) \hookrightarrow Cl_p(G, C_0). \quad (2.1)$$

The ring D is local with maximal ideal $(p) \subset D$.

Given $H \subseteq G$, there is a transfer map $K_p^0(BH) \rightarrow K_p^0(BG)$, which is a map of modules for the $K_p^0(BG)$ -module structure coming from restriction. This transfer map is compatible with the transfer map on class functions

$$\text{Tr}_H^G: Cl_p(H, C_0) \rightarrow Cl_p(G, C_0)$$

given by

$$\text{Tr}_H^G(f)([g]) = \frac{1}{|H|} \sum_{k \in G, k g k^{-1} \in H} f(k g k^{-1}),$$

see for example [HKR00, Theorem D]. Given a finite abelian p -group A and a subgroup $A' \subseteq A$, the transfer map $K_p^0(BA') \rightarrow K_p^0(BA)$ is determined by the image of $1 \in K_p^0(BA')$ as the restriction $K_p^0(BA) \rightarrow K_p^0(BA')$ is surjective. Applying the formula above, we have

$$\text{Tr}_{A'}^A(1)(a) = \begin{cases} |A/A'| & \text{if } a \in A' \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

We define $I_{tr} \subset K_p^0(BA)$ to be the ideal generated by the image of the transfer maps along all proper subgroups of A . We will abuse notation and denote the ideal generated by transfers along all proper subgroups in $Cl_p(A, C_0)$ by I_{tr} as well. In fact, since the class functions in (2.2) take values in the integers, we may also consider the ideal I_{tr} in $Cl_p(A, D)$.

Lemma 2.3. *Assume A is a finite abelian p -group, then the commutative ring $K_p^0(BA)/I_{tr}$ is non-zero.*

Proof. This is clear when A is cyclic. Now we assume that $\text{rank}(A) \geq 2$. Since there is a map of commutative rings

$$K_p^0(BA)/I_{tr} \rightarrow Cl_p(A, D)/I_{tr} \cong \left(\prod_{a \in A} D \right) / I_{tr},$$

it suffices to prove that the target is non-zero. Consider the factor of $\prod_{a \in A} D$ corresponding to $0 \in A$. By the formula in (2.2), the projection of I_{tr} to that factor is contained in the ideal $(p) \subset D$. \square

Lemma 2.4. *The commutative ring $K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr}$ is an \mathbb{F}_p -algebra.*

Proof. By Lemma 2.3, it suffices to show that $(p) \subseteq I_{tr}$. Let V_1, \dots, V_{p+1} be the maximal abelian subgroups of $A = (\mathbb{Z}/p)^{\times 2}$, then

$$p = \left(\sum_{i=1}^{p+1} \text{Tr}_{V_i}^A(1) \right) - \text{Tr}_0^A(1).$$

This can be checked in the ring of class functions. The value of the class function

$$\sum_{i=1}^{p+1} \text{Tr}_{V_i}^A(1)$$

is p on any non-zero element of A and $p^2 + p$ on $0 \in A$. \square

Proposition 2.5. *Assume A is a finite abelian p -group with $\text{rank}(A) \geq 2$, then the commutative ring $K_p^0(BA)/I_{tr}$ is an \mathbb{F}_p -algebra.*

Proof. Fix a surjection $\rho: A \twoheadrightarrow (\mathbb{Z}/p)^{\times 2}$. Since ρ is surjective, given any proper subgroup $H \subset (\mathbb{Z}/p)^{\times 2}$, $\rho^{-1}(H)$ is a proper subgroup of A . Thus the map ρ induces a map of commutative rings

$$K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr} \rightarrow K_p^0(BA)/I_{tr}.$$

It suffices to prove that $K_p^0(B(\mathbb{Z}/p)^{\times 2})/I_{tr}$ is an \mathbb{F}_p -algebra, which is the content of Lemma 2.4. \square

3. MORAVA E -THEORY OF FINITE ABELIAN p -GROUPS AND TORSION

Let E be a height n Morava E -theory at the prime p and let A be a finite abelian p -group. Just as before, we define the transfer ideal $I_{tr} \subset E^0(BA)$ to be the ideal generated by the image of the transfer maps along proper subgroups of A .

Morava E -theory admits a variety of versions of character theory (e.g., [HKR00], [GS99], [Sta13], [Sta15], [BS16]). In [BS16, Section 5] the authors constructed a commutative ring \bar{C}_1 , which depends on n and p , that is a flat E^0 -algebra and a faithfully flat \mathbb{Z}_p -algebra with the property that

$$\bar{C}_1 \otimes_{E^0} E^0(BA) \cong \prod_{A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA).$$

In fact, the isomorphism above can be extended to all finite groups but we will not need that generality here. Write \mathcal{L} for the free loop space functor. Since $\mathcal{L}^{n-1}BA \simeq A^{\times n-1} \times BA$, we have an isomorphism

$$\prod_{A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA) \cong \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA).$$

The isomorphism is compatible with transfers in the sense that there is a commutative diagram of abelian groups

$$\begin{array}{ccc} E^0(BA') & \longrightarrow & \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA') \\ \text{Tr} \downarrow & & \downarrow \bar{C}_1 \otimes_{\mathbb{Z}_p} \text{Tr} \\ E^0(BA) & \longrightarrow & \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(\mathcal{L}^{n-1}BA), \end{array}$$

where the transfer on the right is along the finite cover $\mathcal{L}^{n-1}BA' \rightarrow \mathcal{L}^{n-1}BA$. The commutativity of this diagram is due to the fact that it is induced by a map of cohomology theories. This is explained in Proposition 4.14 in [BS17].

We say that a commutative ring R contains torsion if the underlying \mathbb{Z} -module contains torsion. For a flat map $R \rightarrow S$ of commutative rings, an R -module M contains p -torsion if $S \otimes_R M$ contains p -torsion. This follows from the fact that if $p: M \rightarrow M$ is injective then $p: S \otimes_R M \rightarrow S \otimes_R M$ is injective.

Theorem 3.1. *Assume that A is an abelian p -group with $\text{rank}(A) \geq 2$, then the commutative ring $E^0(BA)/I_{tr}$ contains p -torsion.*

Proof. It follows from Theorem 6.9 in [BS16] that

$$\bar{C}_1 \otimes_{E^0} E^0(BA)/I_{tr} \cong \prod_{(a_i) \in A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA)/I_{tr}^{(a_i)},$$

where $I_{tr}^{(a_i)}$ is a certain transfer ideal depending on the tuple (a_i) . The transfer ideals in the target were studied in [HS20]. The ideal $I_{tr}^{(0)} \subset K_p^0(BA)$ is the ideal generated by transfers from all proper subgroups of A . Proposition 2.5 implies that $K_p^0(BA)/I_{tr}^{(0)}$ has torsion. Since \bar{C}_1 is faithfully flat as a \mathbb{Z}_p -algebra, the ring

$$\prod_{(a_i) \in A^{\times n-1}} \bar{C}_1 \otimes_{\mathbb{Z}_p} K_p^0(BA)/I_{tr}^{(a_i)}$$

contains torsion, hence so does $\bar{C}_1 \otimes_{E^0} E^0(BA)/I_{tr}$. Since \bar{C}_1 is a flat E^0 -algebra, it follows that $E^0(BA)/I_{tr}$ contains torsion as well. \square

We conclude with a short comparison of Theorem 3.1 with the existing literature. In [Str98], Strickland proves that $E^0(B\Sigma_{p^k})/I_{tr}$ is a free E^0 -module, where I_{tr} is the ideal generated by transfers along $\Sigma_i \times \Sigma_j \subset \Sigma_{p^k}$ with $i + j = p^k$ and $i, j > 0$. Of course, Σ_{p^k} is only abelian when $p = 2$ and $k = 1$, in which case it is cyclic. Further, given a transitive abelian subgroup $A \subseteq \Sigma_{p^k}$, there is an induced map of commutative rings

$$E^0(B\Sigma_{p^k})/I_{tr} \rightarrow E^0(BA)/I_{tr}.$$

However, due to the direction of the map, Theorem 3.1 does not have any consequences on the existence torsion in $E^0(B\Sigma_{p^k})/I_{tr}$.

In [SS15], the main result of [Str98] is generalized to groups of the form $A \wr \Sigma_{p^k}$. The transfer ideal $I_{tr} \subset E^0(BA \wr \Sigma_{p^k})$ considered there is generated by transfers along $A \wr (\Sigma_i \times \Sigma_j) \subseteq A \wr \Sigma_{p^k}$ with $i + j = p^k$ and $i, j > 0$. They show that $E^0(BA \wr \Sigma_{p^k})/I_{tr}$ is a free E^0 -module. In [Nel18], Nelson proves that $E^0(B\Sigma_{p^k} \wr \Sigma_{p^l})/I_{tr}$ is free, where I_{tr} is generated by transfers along the subgroups $(\Sigma_i \times \Sigma_j) \wr \Sigma_{p^l} \subset \Sigma_{p^k} \wr \Sigma_{p^l}$, where $i + j = p^k$ and $i, j > 0$ and along the subgroups $\Sigma_{p^k} \wr (\Sigma_i \times \Sigma_j) \subset \Sigma_{p^k} \wr \Sigma_{p^l}$, where $i + j = p^l$ and $i, j > 0$. In both of these cases, these results do not contradict our main result.

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