§14.6 Directional Derivatives

def. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

(if this limit exists)

Special cases

$\vec{u} = \langle 1, 0 \rangle = \vec{i}$

$$D_{\vec{i}} f = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial x}$$

$\vec{u} = \langle 0, 1 \rangle = \vec{j}$

$$D_{\vec{j}} f = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} = \frac{\partial f}{\partial y}$$

Theorem: If f is a differentiable fn. of $x + y$, then f has a directional deriv. in the direction of any unit vector $\vec{u} = \langle a, b \rangle$ with

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y) \cdot b.$$

$$= \langle f_x, f_y \rangle \cdot \langle a, b \rangle.$$

<u>Proof</u>

let $g(h) = f(x_0 + ha, y_0 + hb)$.

Then

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0).$$

By chain rule ($g(h) = f(x,y)$ with $x = x_0 + ha$, $y = y_0 + hb$)

$$g'(h) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dh} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dh},$$

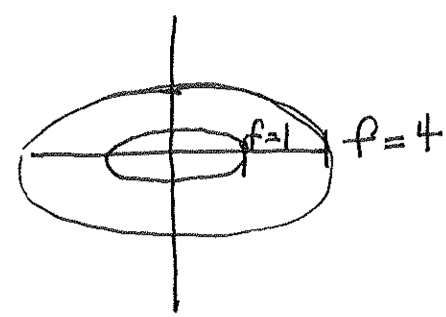
$$= f_x^{(x,y)} \cdot a + f_y^{(x,y)} \cdot b.$$

so

$$g'(0) = f_x(x_0, y_0)\, a + f_y(x_0, y_0)\, b,$$

$$= D_{\vec{u}} f(x_0, y_0) \qquad \boxed{S}$$

ex. $f(x,y) = x^2 + 4y^2$.



$\vec{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$.

$f_x = 2x \qquad f_y = 8y.$

$$D_{\vec{u}} f(x,y) = \langle f_x, f_y \rangle \cdot \vec{u}$$

$$= \frac{2x}{\sqrt{2}} + \frac{8y}{\sqrt{2}}.$$

when $(x,y) = (1,0)$.

$$D_{\vec{u}}(1,0) = \frac{2}{\sqrt{2}}.$$

Theorem: f diff'ble fn.
Then the max value of
$D_{\vec{u}} f(\vec{x})$ is $|\nabla f(\vec{x})|$
& it occurs when
$\vec{u}$ has same direction as
$\nabla f(\vec{x})$.

<u>Proof</u>

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$
$$= |\nabla f| \underbrace{|\vec{u}|}_{1} \cos\theta$$
$$= |\nabla f| \underbrace{\cos\theta}_{\text{max when } = 1}$$

or

$\theta = 0$.

i.e., ∇f & $\vec{u}$ point in same direction.

$\boxed{5}$

Cor: Min value: ∇f & $\vec{u}$ point in opposite directions!

ex. Walking down a mountain $x^2+y^2 = f(x,y)$