Math 213 - Geometry of Space Curves

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September 14, 2018

Homework

- Your first Exam is Wednesday, September 19
- Re-re-read section 13.3, pp. 861–867
- Begin working on pp. 868–870, 1, 3, 5, 11, 13, 17-27 (odd), 47, 49
- Read section 13.4, pp. 870–877
- Remember that homework A5 is due Exam Wednesday, but you might want to finish it before then!

Unit I: Geometry and Motion in Space

Lecture 1	Three-Dimensional Coordinate Systems
Lecture 2	Vectors
Lecture 3	The Dot Product
Lecture 4	The Cross Product
Lecture 5	Equations of Lines and Planes, Part I
Lecture 6	Equations of Lines and Planes, Part II
Lecture 7	Cylinders and Quadric Surfaces
Lecture 8	Vector Functions and Space Curves
Lecture 9	Derivatives and Integrals of Vector Functions
Lecture 10	Arc Length and Curvature
Lecture 11	Motion in Space: Velocity and Acceleration
Lecture 12	Exam 1 Review

Goals of the Day

- Know how to visualize and compute the curvature $\kappa(t)$ of a space curve at a given point $\mathbf{r}(t)$
- Know how to visualize and compute the normal vector $\mathbf{N}(t)$ and the binormal vector $\mathbf{B}(t)$ for a space curve
- Know how to visualize and compute the *osculating circle* and the *osculating plane* for a space curve at a given point $\mathbf{r}(t)$

Remember: Visualize *and* compute!

A Stray Fact for Later Use

Recall that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

so that

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1$$

Taking the derivative of both sides we discover that

$$\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$$

so that $\mathbf{T}'(t)$ is always orthogonal to $\mathbf{T}(t)$!

Parameterizing by Arc Length

Suppose C is the space curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ for $a \leq t \leq b$. The total arc length is

$$L = \int_a^b \left| \mathbf{r}'(u) \right| \, du$$

The arc length traveled from $t = a$ to t is

$$s(t) = \int_a^t \left| \mathbf{r}'(u) \right| \, du$$

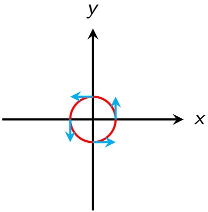
We can reparametrize the curve C using the arc length parameter s, $0 \leq s \leq L$.

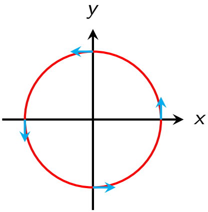
Parameterize the curve

$$\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), t \rangle$$

by arc length.

Curvature - Example





1. $\mathbf{r}(s) = \langle \cos(s), \sin(s) \rangle$ has unit tangent

$$\mathbf{T}(s) = \langle -\sin(s), \cos(s) \rangle$$

and

$$\frac{d\mathbf{T}}{ds} = \langle -\cos(s), -\sin(s) \rangle$$

2. $\mathbf{r}(s) = \langle 3\cos(s/3), 3\sin(s/3) \rangle$ has unit tangent

$$\mathbf{T}(s) = \langle -\sin(s/3), \cos(s/3) \rangle$$

and

$$\frac{d\mathbf{T}(s)}{ds} = (1/3)\langle -\cos(s/3), -\sin(s/3) \rangle$$

Moral: The quantity $|d\mathbf{T}(s)/ds|$ measures *curvature*

Curvature - Definition

The **curvature** of a space curve is given by

$$\kappa(s) = \left| \frac{d\mathbf{T}(s)}{ds} \right|$$

---

1. Find the curvature of the curve $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$
2. Find the curvature of the curve $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle$

Moral: The curvature is the reciprocol of the radius of the "best fitting" circle

Curvature - Other Formulas

Recall

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du$$

so by the Fundamental Theorem of Calculus

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

Curvature - Other Formulas

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Remembering the chain rule, this means we can compute

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds} = \frac{1}{ds/dt} \frac{d\mathbf{T}}{dt}$$

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or

$$\frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|}$$

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or

$$\frac{d\mathbf{T}}{ds} = \frac{\mathbf{T}'(t)}{|\mathbf{r}'(t)|}$$

so that

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Curvature - Other Formulas

A yet more user-friendly formula is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

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Step 1: $\mathbf{r}'(t) = \dfrac{ds}{dt}\mathbf{T}(t)$

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Step 2: $\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$

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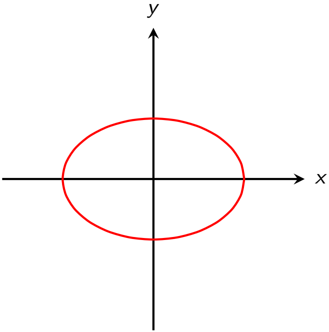
Step 4: $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \left(\frac{ds}{dt}\right)^2 |\mathbf{T}'(t)|$

Step 5: $\kappa(t) = \frac{|\mathbf{T}'(t)|}{ds/dt} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{(ds/dt)^3}$

Curvature - The Ellipse



$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

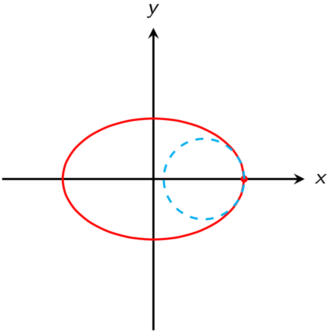
For the ellipse

$$\mathbf{r}(t) = \langle a \cos t, b \sin t \rangle$$

we get

$$\kappa(t) = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$$

Curvature - The Ellipse



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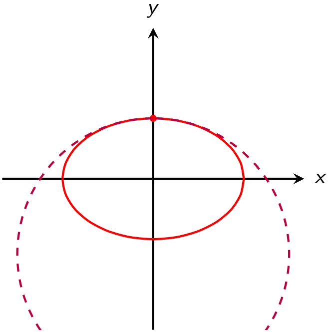

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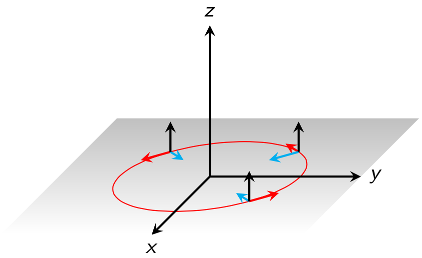
we get

$$\kappa(t) = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$$

- $\kappa(0) = ab/b^3$
- $\kappa(\pi/2) = ab/a^3$

The **T**, **N**, and **B** Vectors

- Unit tangent vector $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
- Unit normal vector $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
- Unit binormal vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$



If $\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$, these vectors are

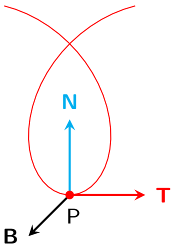
$$\mathbf{T}(t) = \langle -\sin(t), \cos(t), 0 \rangle$$
$$\mathbf{N}(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$
$$\mathbf{B}(t) = \langle 0, 0, 1 \rangle$$

T points along the curve

N points along the curvature

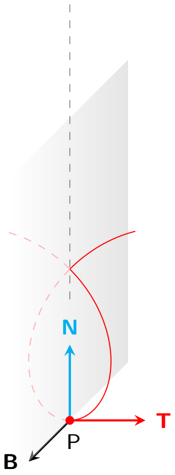
B is orthogonal to **T** and **N**

Normal Plane, Osculating Plane, Osculating Circle



P is a point on the space curve C.

Normal Plane, Osculating Plane, Osculating Circle



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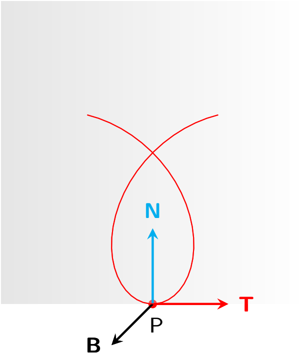
normal plane - spanned by **N** and **B**

Normal Plane, Osculating Plane, Osculating Circle



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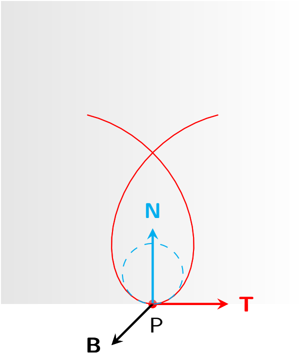
osculating plane - spanned by **N** and **T**

Normal Plane, Osculating Plane, Osculating Circle



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normal plane - spanned by **N** and **B**

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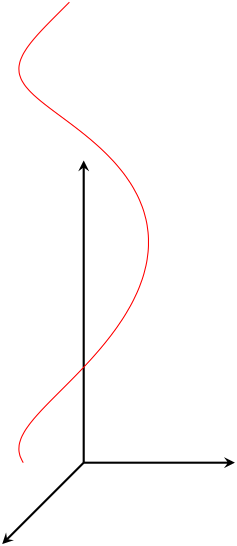
osculating circle

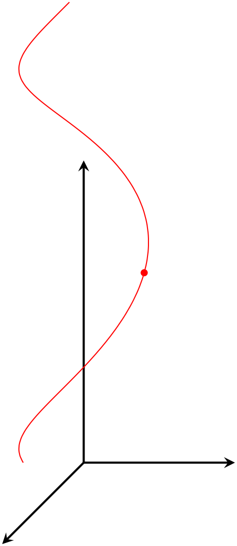
- radius $1/\kappa(P)$,
- center in **N** direction

Normal Plane, Osculating Plane



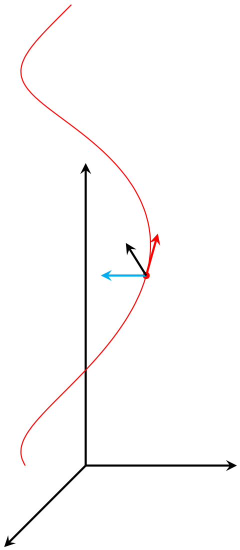
Find the equations of the normal and osculating plane of $\mathbf{r}(t) = \langle 2\sin 2t, -2\cos 2t, 4t\rangle$ at $t = \pi/2$

Normal Plane, Osculating Plane



$$\mathbf{r}(t) = \langle 2\sin 2t, -2\cos 2t, 4t\rangle$$
$$\mathbf{r}(\pi/2) = \langle 0, 2, 2\pi\rangle$$

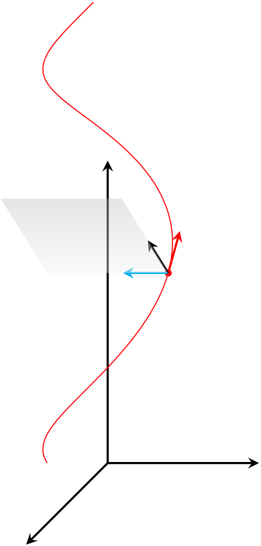
Normal Plane, Osculating Plane



$$\mathbf{r}(t) = \langle 2\sin 2t, -2\cos 2t, 4t\rangle$$
$$\mathbf{r}(\pi/2) = \langle 0, 2, 2\pi\rangle$$

$$\mathbf{T}(\pi/2) = \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle$$
$$\mathbf{N}(\pi/2) = \langle 0, -1, 0\rangle$$
$$\mathbf{B}(\pi/2) = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$$

Normal Plane, Osculating Plane



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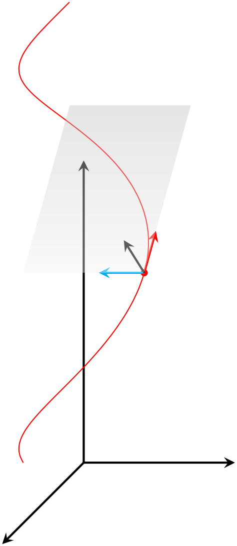
Normal plane: $-x + z = 2\pi$

Normal Plane, Osculating Plane



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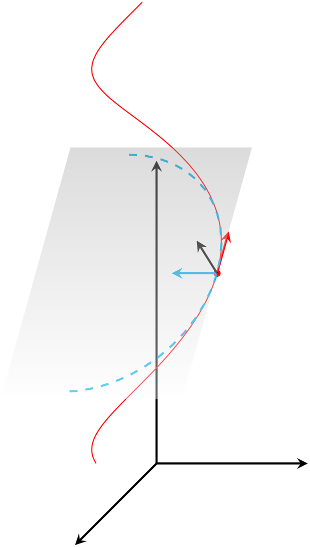
Normal plane: $-x + z = 2\pi$

Osculating plane: $x + z = 2\pi$

Osculating Circle



$$\mathbf{r}(t) = \langle 2\sin 2t, -2\cos 2t, 4t \rangle,$$

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Curvature

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$$\kappa(\pi/2) = 1/4$$

Osculating circle has radius 4