Math 213 - Motion in Space

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Homework

- Your first exam is Wednesday at 5:00 PM - Know your room in CB!
- Homework A5 is due Wednesday night
- You should re-read section 13.4 to prepare for tomorrow's recitation
- You should begin work on Stewart problems 3, 7, 9, 11, 23, 25, 37, 39, 41 from section 13.4
- You should begin reviewing for your exam!

Unit I: Geometry and Motion in Space

Goals of the Day

- Know how to compute velocity and acceleration
- Know how to solve projectile problems
- Understand tangential and normal components of acceleration
- Understand Kepler's Laws of Planetary Motion

Velocity and Acceleration

If $\mathbf{r}(t)$ is the space curve of a moving body and if t is time:

- $\mathbf{r}'(t)$ is $\mathbf{v}(t)$, the *velocity* of the moving body
- $|\mathbf{r}'(t)|$ is the *speed* of the moving body
- $\mathbf{r}''(t)$ is $\mathbf{a}(t)$, the *acceleration* of the moving body

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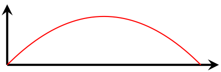
1. (Projectile motion) Suppose that $\mathbf{r}(t) = \langle 32t, 32t - 16t^2 \rangle$. Find the velocity and acceleration
2. (Circular motion) Suppose that $\mathbf{r}(t) = \langle R\cos(2\pi t/T), R\sin(2\pi t/T) \rangle$. Find the velocity and acceleration.

Velocity and Acceleration



$\mathbf{r}(t) = \langle 32t, 32t - 16t^2 \rangle$.

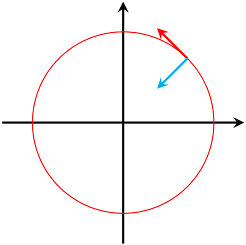
What's the projectile's acceleration?

When does the projectile hit the ground?

What is its speed when it hits?

How far does it go?

What is its maximum height?



$\mathbf{r}(t) = \langle R\cos(2\pi t/T), R\sin(2\pi t/T) \rangle$

How long does one orbit take?

Which way does the velocity vector point?

Which way does the acceleration vector point?

Interlude - Newton's Laws of Motion

1. A body will remain at rest or in motion in a straight line unless acted on by an external force.
2. The applied force $\mathbf{F}$ is equal to the change of momentum $m\mathbf{v}$ per unit time
3. For every action there is an equal and opposite reaction

Projectile Motion

For constant mass, Newton's second law implies

$$\mathbf{F} = m\mathbf{a}$$

(Warning: Do not use for rockets!)

At the surface of the earth, a mass m is subject to a gravitational force $-mg\mathbf{j}$

From Newton's second law we then get $m\mathbf{a} = -mg\mathbf{j}$ or

$$\mathbf{r}''(t) = \mathbf{a} = -g\mathbf{j}$$

where $g = 32\ \text{ft}/\text{sec}^2 = 9.8\ \text{m}/\text{sec}^2$.

If we know the *initial conditions* for a projectile (its position and velocity at time zero), we can integrate this equation to find the motion of the projectile.

Projectile Motion

A ball is thrown at an angle of 45° to the ground. If the ball lands 90 m away, what was the initial speed of the ball?

$$\begin{aligned}
\mathbf{r}''(t) &= -9.8\mathbf{k} \\
\mathbf{r}'(0) = \mathbf{v}(0) &= v_0 \cos(45^\circ)\mathbf{i} + v_0 \sin(45^\circ)\mathbf{j} \\
\mathbf{r}(0) &= 0\mathbf{i} + 0\mathbf{j}
\end{aligned}$$

Now integrate:

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^t \mathbf{a}(s)\, ds \\
&= v_0(\sqrt{2}/2)\mathbf{i} + \left(v_0(\sqrt{2}/2) - 9.8t\right)\mathbf{j} \\
\mathbf{r}(t) &= \mathbf{r}(0) + (v_0(\sqrt{2}/2)t)\mathbf{i} + \left(v_0(\sqrt{2}/2)t - (9.8/2)t^2\right)\mathbf{j}
\end{aligned}$$

Now what?

Components of Acceleration

Suppose $\mathbf{r}(t)$ is the path of a moving object.

Let's abbreviate $\mathbf{v}(t) = \mathbf{r}'(t)$, $v(t) = |\mathbf{r}'(t)|$. Then

$$\mathbf{v}(t) = v(t)\mathbf{T}(t)$$

Recall that

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)}$$

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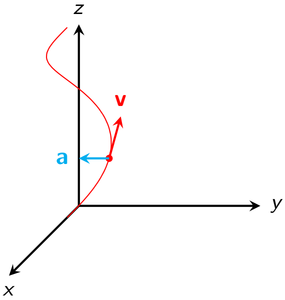
The acceleration of the object is

$$\begin{aligned}
\mathbf{a}(t) &= \frac{d}{dt} v(t)\mathbf{T}(t) \\
&= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \\
&= v'(t)\mathbf{T}(t) + v(t)|\mathbf{T}'(t)|\mathbf{N}(t) \\
&= v'(t)\mathbf{T}(t) + \kappa(t)v(t)^2\mathbf{N}(t)
\end{aligned}$$

That is, roughly,

$$\text{total acceleration} = \text{change of speed } \mathbf{T} + \text{curvature } v^2\mathbf{N}$$

Components of Acceleration



$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

Find tangential and normal components of acceleration for the path

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

$$\begin{aligned} v(t) &= \sqrt{2} \\ \mathbf{T}(t) &= \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle \\ \mathbf{N}(t) &= \langle -\cos(t), -\sin(t), 0 \rangle \\ \kappa(t) &= 1/4 \end{aligned}$$

Kepler's Laws

1. A planet orbits around the sun in an elliptical orbit with the sun at one focus
2. The line joining the sun to the planet sweeps out equal areas in equal times
3. The square of the period T of a planet's orbit is proportion to the cube of the major axis a of the ellipse

In Newton's *Principia Mathematica*, Newton showed how to derive Kepler's laws from:

Newton's Second Law: $\mathbf{F} = m\mathbf{a}$

Newton's Law of Gravitation: $\mathbf{F} = -\dfrac{GMm}{r^3}\mathbf{r}$

Planetary Orbits are Planar

Consider the vector $\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{v}(t)$. We'll show that this vector is *constant* for a planet moving around the sun, which implies that a planet's orbit lies on a plane.

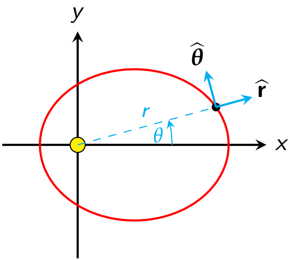
$$\begin{aligned} \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) &= \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a} \\ &= \mathbf{0} + \mathbf{r} \times \left(-\frac{GMm}{r^3}\mathbf{r}\right) \\ &= \mathbf{0} \end{aligned}$$

The constant vector $\mathbf{L}$ defines a plane containing $\mathbf{r}$ and $\mathbf{v}$ since $\mathbf{h} \cdot \mathbf{r} = \mathbf{h} \cdot \mathbf{v} = 0$

The quantity $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ is called *angular momentum* and is conserved for motion governed by a central force.

The Equal Areas Law

Recall $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ is conserved, and that the area swept out by a polar curve is $A = \frac{1}{2}\int_a^b r^2\, d\theta$



Introduce polar coordinates (r, θ) and let

$$\widehat{\mathbf{r}} = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$$

$$\widehat{\boldsymbol{\theta}} = -\sin\theta\, \mathbf{i} + \cos\theta\, \mathbf{j}$$

Then

$$\mathbf{r} = r\,\widehat{\mathbf{r}}$$

$$\mathbf{v} = \dot{r}\,\widehat{\mathbf{r}} + r\frac{d}{dt}\left(\widehat{\mathbf{r}}\right) = \dot{r}\,\widehat{\mathbf{r}} + r\dot{\theta}\,\widehat{\boldsymbol{\theta}}$$

so

$$\mathbf{r} \times m\mathbf{v} = m\left(r\widehat{\mathbf{r}}\right) \times \left(\dot{r}\,\widehat{\mathbf{r}} + r\dot{\theta}\,\widehat{\boldsymbol{\theta}}\right) = mr^2\dot{\theta}\left(\widehat{\mathbf{r}} \times \widehat{\boldsymbol{\theta}}\right)$$

Since $\mathbf{L}$ is constant, so also $mr^2\dot{\theta}$ is constant