§14.5 Chain Rule.

Calc I. (again).

$$y = f(x)$$
$$x = g(t).$$

where f + g are differentiable then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Chain Rule (part 1): $z = f(x,y)$ a differentiable function of x + y, $x = g(t)$, $y = h(t)$ for g + h diffble fns of t.

Then z is a diffble fn of t with

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

Tree diagram

$$\begin{tikzcd}
 & z \arrow[dl, dash] \arrow[dr, dash] & \\
x \arrow[d, dash] & & y. \arrow[d, dash] \\
t & & t.
\end{tikzcd}$$

Proof

Recall def'n of differentiable.

Informally, (This just says plane tangent line approx z well).

$$\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$

where $(\Delta x, \Delta y) \to (0,0)$
as $\epsilon_1, \epsilon_2 \to 0$.

Divide by Δt. Take limit as $\Delta t \to 0$.

$$\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \lim_{\Delta t \to 0} f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \epsilon_1 \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = f_x\, dx + f_y\, dy + \lim_{\Delta t \to 0} \epsilon_1 \cdot \frac{\Delta x}{\Delta t} + \epsilon_2 \frac{\Delta y}{\Delta t}.$$

$\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}$, $\frac{\Delta y}{\Delta t} \to \frac{dy}{dt}$ since x & y are dfble wrt t.

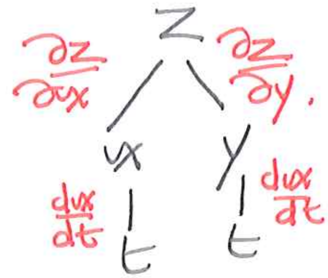
$\epsilon_1 \to 0$, $\epsilon_2 \to 0$ since $\Delta x \to 0$ & $\Delta y \to 0$ as $\Delta t \to 0$.

Find $\frac{dz}{dt}$.

ex. $z = \sqrt{x^2+y^2}$

$x = \cos t$

$y = \sin t$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x \ (-\sin t) \ +$$
$$\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y \ (\cos t)$$

$$= \frac{1}{2} \cdot 1 \cdot 2\cos t\,(-\sin t) +$$
$$\frac{1}{2} \cdot 1 \cdot 2\sin t\,(\cos t) = 0 \qquad \text{(check!)}$$

ex. $z = \tan^{-1}(x \cdot y)$ Find $\frac{dz}{dt}$ @ $t=0$.

$x = \tan t$

$y = e^t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{1+(xy)^2} \cdot y \cdot \sec^2 t + \frac{1}{1+(xy)^2} \cdot x \cdot e^t$$

When $t=0$,

$x = \tan 0 = 0$

$y = e^0 = 1$

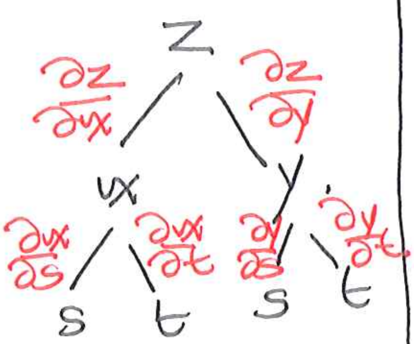
$$\Rightarrow \frac{dz}{dt} = \frac{1}{1+0^2} \cdot 1 \cdot \sec^2 0 + 1 \cdot 0 \cdot e^0$$
$$= 1 + 0 = 1.$$

Chain Rule in 3 variables:

$z = f(x,y)$ dfble fn of x & y

$x = g(s,t)$, $y = h(s,t)$ dfble fns of s & t.

Then,



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}.$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Adds in more variables!

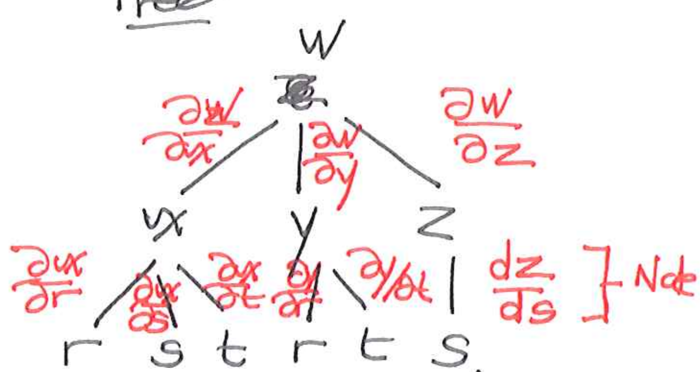
ex. $\cancel{z}\, w = x^2y^3z$.

$x = 3rst$

$y = r^2t$

$z = s^3$.

Tree



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial t} + \cancel{\partial w}$$

z does not depend on t

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \cancel{\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s}} + \frac{\partial w}{\partial z}\cdot\frac{dz}{ds}$$

y does not depend on s

(The dreaded)

Implicit Differentiation.

ex. Find $\frac{dy}{dx}$ of $x^2+6xy=5y^2-3$

<u>Soln</u>

Rewrite as $F(x,y)=0$.

$$\underbrace{x^2+6xy-5y^2+3}_{F(x,y).}=0$$

This defines y implicitly as a differentiable function of x.

Differentiate $F(x,y)=0$ wrt x.

$$\underbrace{\frac{\partial F}{\partial x}}_{\cancel{\text{Assume}\neq 0}}\cdot\underbrace{\frac{dx}{dx}}_{1.}+\underbrace{\frac{\partial F}{\partial y}}_{\text{Assume nonzero}}\cdot\frac{dy}{dx}=0.$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}.}=\ominus\,\frac{F_x}{F_y}.$$

This is the
Implicit Function Theorem
of Advanced Calculus.

If,

① F defined on an open disk containing
point (a,b).

② $F(a,b) = 0$

③ $F_y(a,b) \neq 0$.

④ F_x + F_y continuous on disk.

then

$F(x,y)=0$ defines y as a function of
x near the point (a,b)

+,

$$\frac{dy}{dx} = - \frac{F_x}{F_y}.$$

ex. (return to).

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{(2x+6y)}{(6x-10y)} = \frac{-(x+3y)}{(3x-5y)}.$$

Implicit Function Thm
for $F(a, b, c)$ ...