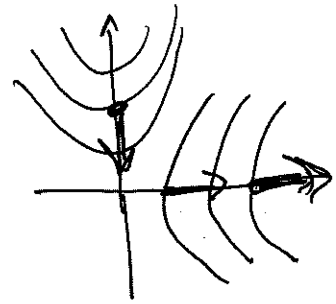
① $f(x,y) = x^2 - y^2$

$x^2 - y^2 = k^2$



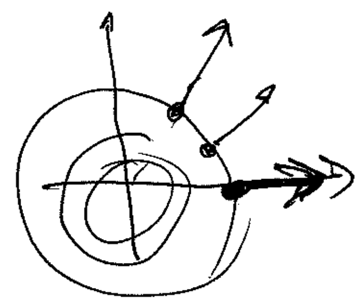
$\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla f(x,0) = \langle 2x, 0 \rangle$

$\nabla f(0,y) = \langle 0, -2y \rangle$

② $g(x,y) = x^2 + y^2$

$\nabla g(x,y) = \langle 2x, 2y \rangle$



Parallel gradients!

x- and y-axis

$f(x,y) \qquad x(t),\ y(t)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \left\langle \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right\rangle = 0$$

$\nabla f \qquad$ tangent

① Max/Min $f(x,y) = x^2 - y^2$

subject to $g(x,y) = x^2 + y^2 = 1$

$\nabla f(x,y) = \langle 2x, -2y \rangle$

$\nabla g(x,y) = \langle 2x, 2y \rangle$

$$\therefore \begin{cases} \nabla f = \lambda \nabla g \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} 2x = \lambda \cdot 2x & -(1) \\ -2y = \lambda \cdot 2y & -(2) \\ x^2 + y^2 = 1 & -(3) \end{cases}$$

(1) $x = 0$ or $\lambda = 1$ if $x \neq 0$

(2) If ~~λ≠1~~ $x \neq 0$ (so $\lambda = 1$), then $y = 0$

~~(3)~~ Either: $x = 0$ OR: $y = 0$

(3) If $x = 0$, $y = \pm 1$

If $y = 0$, $x = \pm 1$

x	y	x^2-y^2
0	+1	−1
0	−1	−1
1	0	+1
−1	0	+1

---

(2) $f(x,y) = 3x+y$

Find max/min subject to $x^2+y^2=10$

$f(x,y) = 3x+y$ objective fcn

$g(x,y) = x^2+y^2$ constraint fcn

$\nabla f(x,y) = \langle 3, 1\rangle$ gradients.

$\nabla g(x,y) = \langle 2x, 2y\rangle$

$3 = 2\lambda x$ — (1)

$1 = 2\lambda y$ — (2)

$x^2+y^2 = 10$ — (3)

$(1) \Rightarrow \quad x = \frac{3}{2\lambda}$

$(2) \Rightarrow \quad y = \frac{1}{2\lambda}$

$(3) \Rightarrow \quad \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 10$

$\Rightarrow \quad \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 10$

$\Rightarrow \quad \frac{10}{4\lambda^2} = 10$

$\Rightarrow \quad 4\lambda^2 = 1$

$\lambda^2 = \frac{1}{4}$

$\boxed{\lambda = \pm\frac{1}{2}}$

$\lambda = +\frac{1}{2}$: $\quad x = 3 \quad y = 1$

$\lambda = -\frac{1}{2}$: $\quad x = -3 \quad y = -1$

x	y	$f(x,y) = 3x+y$
3	1	10
-3	-1	-10

Max: 10 at $(3,1)$

Min: -10 at $(-3,-1)$

Find max/min of

$$f(x,y,z) = e^{xyz}$$

subject to the constraint

$$g(x,y,z) = 2x^2+y^2+z^2=24$$

$$\nabla f(x,y,z) = \langle yz e^{xyz}, xz e^{xyz}, xy e^{xyz} \rangle$$

$$\nabla g(x,y,z) = \langle 4x, 2y, 2z \rangle$$

(1) $yz e^{xyz} = 4\lambda x$

(2) $xz e^{xyz} = 2\lambda y$

(3) $xy e^{xyz} = 2\lambda z$

(4) $2x^2+y^2+z^2=24$

(1) $\lambda = \frac{yz}{4x} e^{xyz}$

(2) $\lambda = \frac{xz}{2y} e^{xyz}$

(3) $\lambda = \frac{xy}{2z} e^{xyz}$

$$\frac{yz}{4x} \cancel{e^{xyz}} = \frac{xz}{2y} \cancel{e^{xyz}} = \frac{xy}{2z} \cancel{e^{xyz}}$$

$\frac{y\cancel{z}}{4x} = \frac{x\cancel{z}}{2y}$ $\qquad$ $\frac{\cancel{x}z}{2y} = \frac{\cancel{x}y}{2z}$ $\qquad$ $\frac{\cancel{y}z}{4x} = \frac{x\cancel{y}}{2z}$

$\boxed{2y^2 = 4x^2}$ $\qquad$ $2z^2 = 2y^2$ $\qquad$ $\boxed{2z^2 = 4x^2}$

$$2x^2 + y^2 + z^2 = 24$$

(4) $2x^2 + 2x^2 + 2x^2 = 24$

$$6x^2 = 24$$

$$x^2 = 4$$

$$x = \pm 2$$

So $2y^2 = 16$ $\qquad$ $2z^2 = 4 \cdot 4 = 16$

$y^2 = 8$

$y = \pm 2\sqrt{2}$ $\qquad$ $z = \pm 2\sqrt{2}$

$f(x,y,z) = x+y+z$ objective fcn

$g(x,y,z) = x^2+z^2 = 2$ constraint #1

$h(x,y,z) = x+y = 1$ constraint #2

$\nabla f(x,y,z) = \langle 1, 1, 1 \rangle$

$\nabla g(x,y,z) = \langle 2x, 0, 2z \rangle$

$\nabla h(x,y,z) = \langle 1, 1, 0 \rangle$

(1) $\cancel{1 = 2\lambda x}$

(2) $\cancel{1 = 2\lambda \cdot 0}$

(3) $\cancel{1 = 2\lambda \cdot 2z}$

$\nabla f = \lambda \nabla g + \mu \nabla h$

(

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$$\nabla f = \lambda \nabla g + \mu \nabla h$$

(1) $1 = \lambda \cdot 2x + \mu \cdot 1$

(2) $1 = \lambda \cdot 0 + \mu \cdot 1$

(3) $1 = \lambda \cdot 2z + \mu \cdot 0$

(4) $x^2 + z^2 = 2$

(5) $x + y = 1$

---

(1) $\Rightarrow$ $1 = 2\lambda x + \mu$

(2) $1 = \mu$

(3) $1 = 2\lambda z$

(4) $x^2 + z^2 = 2$

(5) $x + y = 1$