Maximum and Minimum Value Probs.

1) $f(x,y) = x^3 - 6xy + 8y^3$

$f_x(x,y) = 3x^2 - 6y = 3(x^2 - 2y)$

$f_y(x,y) = -6x + 24y^2 = 6(4y^2 - x)$

Find critical pts:

$x^2 - 2y = 0$ — (1)

$4y^2 - x = 0$ — (2)

From (2) $x = 4y^2$. substitute into (1):

$(4y^2)^2 - 2y = 0$ — (3)

$16y^4 - 2y = 0$

$2y(8y^3 - 1) = 0$

$\therefore$ $y = 0$ or $y = \frac{1}{2}$ use (1)

Use (1)

$x^2 = 0$

$x = 0$

$x^2 - 1 = 0$

$x = \pm 1$

$$f_x = 3x^2 - 6y \qquad f_{xx} = 6x$$

$$f_y = -6x + 24y^2 \qquad f_{yy} = 48y$$

$$f_{xy} = -6$$

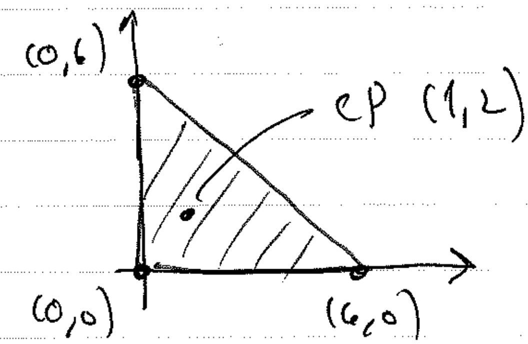
$$\text{Hess}(f) = \begin{pmatrix} 6x & \cancel{48y} \ -6 \\ -6 & 48y \end{pmatrix} \qquad D = 6 \cdot 48xy - 36$$

$$= 36(\cancel{8}\ 8xy - 1)$$

(x,y)	D	$f_{xx}(x,y)$	Type
$(0,0)$	-36		Saddle
$(1, \frac{1}{2})$	$36 \cdot (4-1) > 0$	6	Min
$(-1, \frac{1}{2})$	$36 \cdot (8(-1)(\frac{1}{2}) - 1)$ $= 36 \cdot (-4-1)$ $= 36 \cdot (-5)$ $= -180$		Saddle

2) $f(x,y) = 4xy^2 - x^2y^2 - xy^3$

$D = \Delta$: $(0,0)$, $(0,6)$, $(6,0)$)



$f_x = 4y^2 - 2xy^2 - y^3 \quad = y^2(4 - 2x - y)$

$f_y = 8xy - 2x^2y - 3xy^2 \quad = xy(8 - 2x - 3y)$

Critical Pts:

(1) $y^2(4-2x-y) = 0$

$y = 0 \quad$ OR $\quad y + 2x = 4$

(2) If $y = 0$, <u>any</u> x solves the eq'n

If $y + 2x = 4$: $\quad \boxed{y = 4-2x}$

① ✓ ② ✓ ③

$x(4-2x)(8-2x-3(4-2x)) = 0$

$$\boxed{y = 4-2x}$$

(1) $x=0 \quad y=4$

(2) $x=2 \quad y=0$

(3) $8-2x-12+6x=0 \Rightarrow 4x-4=0$

$x=1,\ y=2$

Critical Pts!

$(x,0)$ cp. ~~$(0,y)$~~ - bdry.

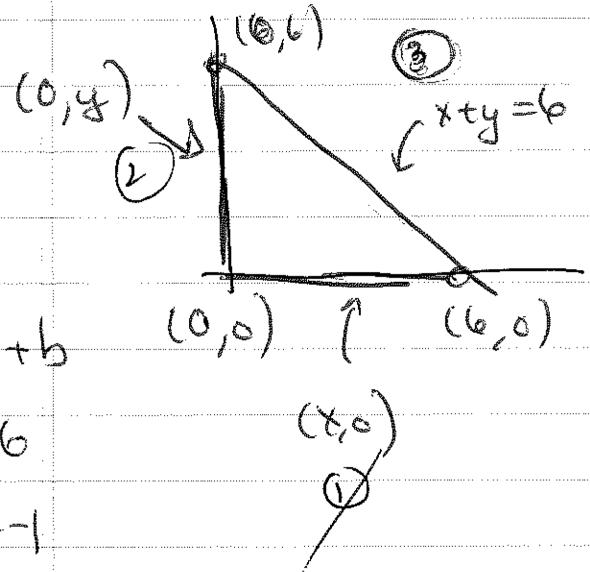
$(0,4)$ - bdry

$(2,0)$ - bdry

$(1,2)$ *

Now consider $f(x,y)$ on the boundary.

$$f(x,y) = 4xy^2 - x^2y^2 - xy^3$$



$f(x,0) = 0$

$f(0,y) = 0$

$y = mx + b$

$b = 6$

$m = -1$

$\varphi(x) = f(x, 6-x) \quad 0 \le x \le 6$

$$\varphi(x) = 4x(6-x)^2 - x^2(6-x)^2 - x(6-x)^3$$

$$= x(6-x)^2 [4 - x - (6-x)]$$

$$= x(6-x)^2 [-2]$$

$$= -2x(6-x)^2$$

$$\varphi'(x) = -2(6-x)^2 - 2x \cdot 2 \cdot (6-x) \cdot (-1)$$

$$= (6-x)[-2(6-x) + 4x]$$

$$= (6-x)[6x - 12]$$

$\varphi'(x) = 0$ at $x = 6,\ x = 2$

Evaluate $\varphi(x) = -2x(6-x)^2$

x	$\varphi(x)$
0	0
2	$-4\cdot(6-2)^2 = \boxed{-64}$
6	$\boxed{0}$

Now check $f(x,y)$ at $(1,2)$

$$f(x,y) = 4xy^2 - x^2y^2 - xy^3$$

$$f(1,2) = 4\cdot 2^2 - 1^2\cdot 2^2 - 1\cdot 2^3$$

$$= 16 - 4 - 8$$

$$= \boxed{4}$$

Max: $\boxed{4}$

Min: $\boxed{-64}$

Lagrange Multiplicos.

1) $f(x,y) = x^2 y$

2) $g(x,y) = x^2+y^2$

$f_x(x,y) = 2xy$ $\qquad$ $g_x(x,y) = 2x$

$f_y(x,y) = x^2$ $\qquad$ $g_y(x,y) = 2y$

So:

$$\nabla f = \lambda \nabla g$$

$$x^2+y^2 = 1$$

(1) $2xy = 2\lambda x$ $\quad \Rightarrow \quad \cancel{x=}\ x=0, \quad \lambda = y$

(2) $x^2 = 2\lambda y$ $\qquad$ (2) $x^2 = 2y^2$

(3) $x^2+y^2 = 1$ $\qquad$ $y = \pm 1$ $\qquad$ (3)

$*\ 2y^2+y^2 = 1$

$3y^2 = 1$

or $y = \pm 1/\sqrt{3}$

$(0, \pm 1)$, $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$

$f(x,y) = x^2 y$

x	y	$x^2 y$
0	1	0
0	-1	0
$\sqrt{\frac{2}{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{3} \cdot \frac{1}{\sqrt{3}}$
$-\sqrt{\frac{2}{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{3} \frac{1}{\sqrt{3}}$
$\sqrt{\frac{2}{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{2}{3} \cdot \frac{1}{\sqrt{3}}$
$-\sqrt{\frac{2}{3}}$	$-\frac{1}{\sqrt{3}}$	$-\frac{2}{3} \frac{1}{\sqrt{3}}$

MAX: $\frac{2}{3\sqrt{3}}$

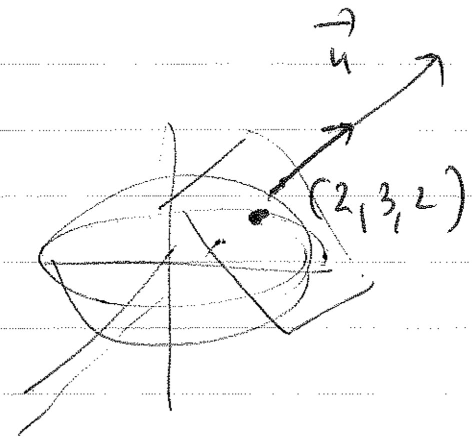
MIN: $-\frac{2}{3\sqrt{3}}$

Normal lines

$f(x, y)$

$f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4}$

$(2, 3, 2)$



$\nabla f(x,y,z) = \langle \frac{x}{2}, \frac{2y}{9}, \frac{z}{2} \rangle$

$\nabla f(2,3,2) = \langle 1, \frac{2}{3}, 1 \rangle$

$x(t) = 2 + t \cdot 1$

$y(t) = 3 + t \cdot \frac{2}{3}$

$z(t) = 2 + t \cdot 1$