Math 213 - Double Integrals Over Rectangles

Peter A. Perry

University of Kentucky

October 17, 2018

Homework

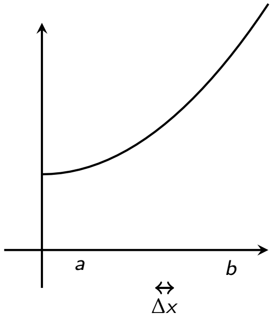
- Ace tonight's exam!
- Be sure to turn in Webwork B5
- Re-read section 15.1
- Begin work on problems 9-21 (odd), 27-43 (odd), 49 from 15.1

Unit III: Multiple Integrals

Goals of the Day

- Understand why the volume under a surface can be computed as a double integral, a limit of double Riemann sums
- Understand how to compute double integrals over rectangles as *iterated integrals*
- Understand how to find the *average value* of a function of two variables over a rectangular domain

Calculus I: Areas under Curves



A Riemann sum with $n = 8$ for $\int_a^b f(x)\,dx$

The area under the graph of $y = f(x)$ between $x = a$ and $x = b$ is approximated by *Riemann sums*

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

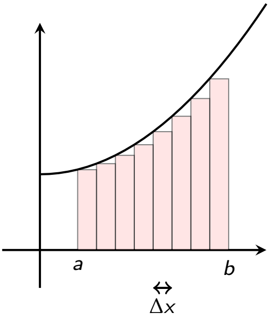
where

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$x_i^* \in [x_{i-1}, x_i]$$

Calculus I: Areas under Curves



A Riemann sum with $n = 8$ for $\int_a^b f(x)\, dx$

The area under the graph of $y = f(x)$ between $x = a$ and $x = b$ is approximated by *Riemann sums*

$$\sum_{i=1}^{n} f(x_i^*)\, \Delta x$$

where

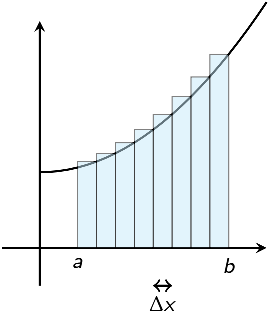
$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$x_i^* \in [x_{i-1}, x_i]$$

$x_i^* = x_{i-1}$ (left endpoint)

Calculus I: Areas under Curves



A Riemann sum with $n = 8$ for $\int_a^b f(x)\,dx$

The area under the graph of $y = f(x)$ between $x = a$ and $x = b$ is approximated by *Riemann sums*

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where

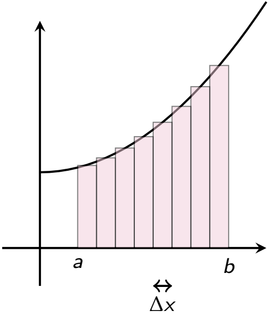
$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$x_i^* \in [x_{i-1}, x_i]$$

$$\begin{aligned} x_i^* &= x_{i-1} && \text{(left endpoint)} \\ x_i^* &= x_i && \text{(right endpoint)} \end{aligned}$$

Calculus I: Areas under Curves



A Riemann sum with $n = 8$ for $\int_a^b f(x)\,dx$

The area under the graph of $y = f(x)$ between $x = a$ and $x = b$ is approximated by *Riemann sums*

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where

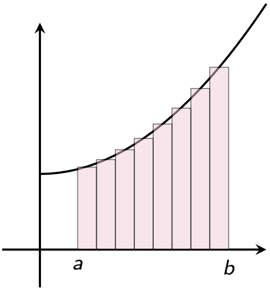
$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$x_i^* \in [x_{i-1}, x_i]$$

$x_i^* = x_{i-1}$	(left endpoint)
$x_i^* = x_i$	(right endpoint)
$x_i^* = \frac{x_{i-1} + x_i}{2}$	(midpoint)

Calculus I: Areas under Curves



The exact area under the graph of $f(x)$ between $x = a$ and $x = b$ is

$$\int_a^b f(x)\,dx$$

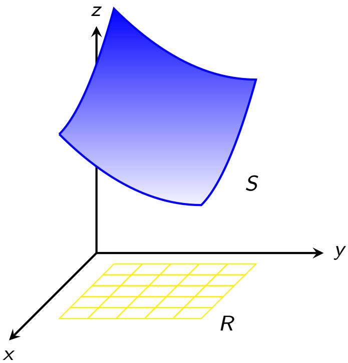
The Fundamental Theorem of Calculus states that, if f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for any antiderivative F of f.

We extended these ideas to compute the *net area* under the graph of a signed function and the *average value* of a function f over an interval $[a, b]$

Calculus II: Volumes under Surfaces



Problem Find the volume between the rectangle

$$R = [a, b] \times [c, d]$$

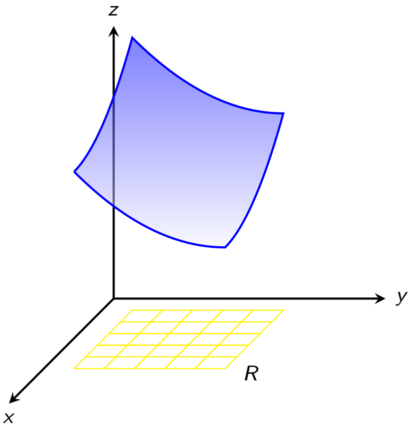
in the xy plane and the surface

$$S = \{(x, y, z) : (x, y) \in R,\ \ z = f(x, y)\}$$

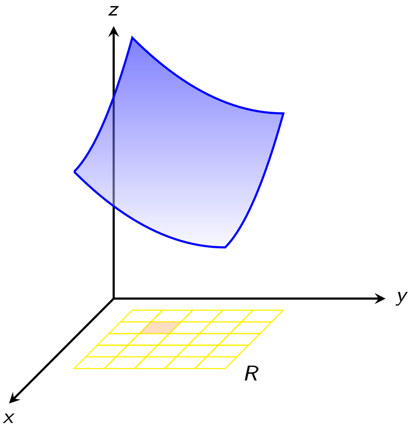
if f is a continuous function.

1. Divide the rectangle R into an $n \times n$ 'grid' of subrectangles $R_{i,j}$
2. For each subrectangle $R_{i,j}$, make a box of height $f(x_i^*, y_j^*)$
3. Add up the volumes of the n^2 boxes

Volumes Under Surfaces

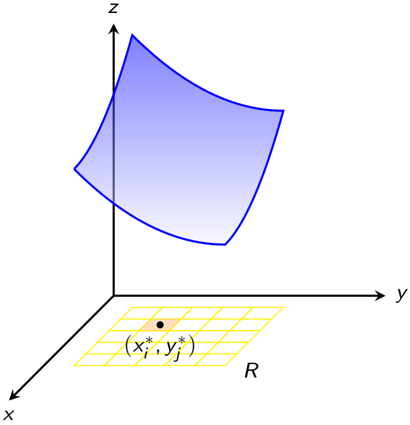


Volumes Under Surfaces



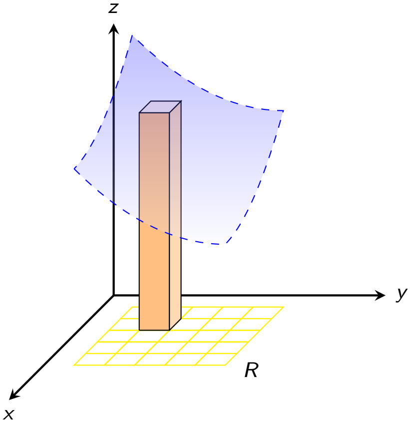
- Pick a rectangle $R_{i,j}$ in the grid, area ΔA

Volumes Under Surfaces



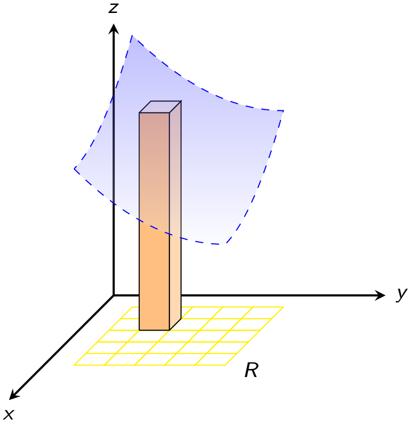
- Pick a rectangle $R_{i,j}$ in the grid, area ΔA
- Pick (x_i^*, y_j^*) in the rectangle

Volumes Under Surfaces



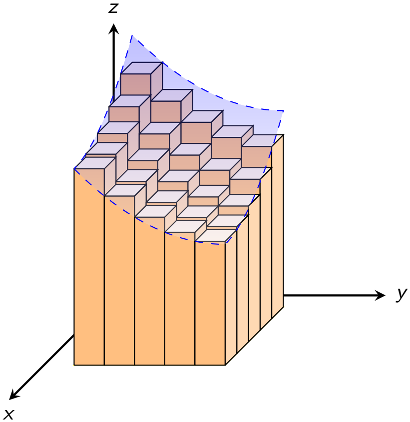
- Pick a rectangle $R_{i,j}$ in the grid, area ΔA
- Pick (x_i^*, y_j^*) in the rectangle
- Make a box of height $f(x_i^*, y_j^*)$ over $R_{i,j}$

Volumes Under Surfaces



- Pick a rectangle $R_{i,j}$ in the grid, area ΔA
- Pick (x_i^*, y_j^*) in the rectangle
- Make a box of height $f(x_i^*, y_j^*)$ over $R_{i,j}$
- The volume of the box is $V_{ij} = f(x_i^*, y_j^*)\,\Delta A$

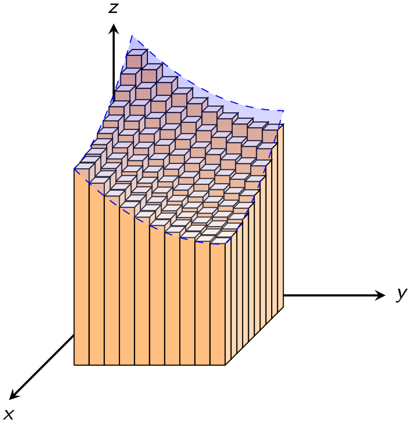
Volumes Under Surfaces



- Pick a rectangle $R_{i,j}$ in the grid, area ΔA
- Pick (x_i^*, y_j^*) in the rectangle
- Make a box of height $f(x_i^*, y_j^*)$ over $R_{i,j}$
- The volume of the box is $V_{ij} = f(x_i^*, y_j^*)\,\Delta A$
- The approximate volume under the surface is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

Volumes Under Surfaces



- Pick a rectangle $R_{i,j}$ in the grid, area ΔA
- Pick (x_i^*, y_j^*) in the rectangle
- Make a box of height $f(x_i^*, y_j^*)$ over $R_{i,j}$
- The volume of the box is $V_{ij} = f(x_i^*, y_j^*)\,\Delta A$
- The approximate volume under the surface is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

$$V = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

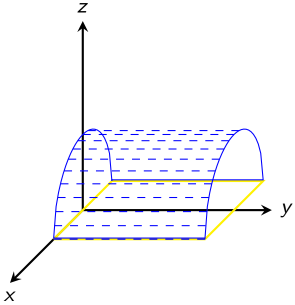
Volumes By Integration

If $R = [a, b] \times [c, d]$ and

$$S = \{(x, y, z) : z = f(x, y),\ (x, y) \in R\}$$

then the volume between R and S is

$$V = \lim_{n\to\infty} \sum_{i,j=1}^{n} f(x_i^*, y_j^*)\,\Delta A = \iint_R f(x, y)\,dA$$



Find $\iint_R f(x, y)\,dA$ if

$$R = [-1, 1] \times [0, 2]$$

and

$$f(x, y) = \sqrt{1 - x^2}$$

Can you do this without calculus?

Iterated Integrals

When you studied one-variable calculus, you first found out how to compute antiderivatives and then you learned how to compute definite integrals using them

Now that you're studying two variable calculus, you'll first learn about *iterated integrals* and then learn how to compute integrals over rectangles with them.

Suppose R is a rectangle $[a, b] \times [c, d]$ and f is a continuous function on R. Then

$$A(x) = \int_c^d f(x, y)\, dy$$

is a function of x. For example if $f(x, y) = x^2 y$ and $R = [1, 2] \times [3, 4]$, then

$$\int_3^4 (x^2 y)\, dy = \left.\frac{x^2 y^2}{2}\right|_3^4 = \frac{7}{2}x^2$$

We then compute $\int_a^b A(x)\, dx$. For example

$$\int_1^2 \frac{7}{2}x^2\, dx = \left.\frac{7}{6}x^3\right|_{x=1}^{x=2} = \frac{49}{6}$$

Iterated Integrals

If $f(x,y)$ is a continuous function on a rectangle $R = [a,b] \times [c,d]$, the **iterated integral** of f is

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right)\,dx$$

---

1. Find $\int_1^4 \int_0^2 \left(6x^2y - 2x\right)\,dy\,dx$
2. Find $\int_1^3 \int_1^5 \frac{\ln y}{xy}\,dy\,dx$
3. (Ringer) $\int_0^1 \int_1^2 \left(x + e^{-y}\right)\,dx\,dy$

Fubini's Theorem

Theorem If f is continuous on the rectangle

$$R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$$

then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Evaluate these double integrals.

1. $\displaystyle\iint_R x \sec^2 y\, dA,\ R = [0, 2] \times [0, \pi/4]$
2. $\displaystyle\iint_R \frac{xy^2}{x^2 + 1}\, dA,\ R = [0, 1] \times [-3, 3]$
3. $\displaystyle\iint_R \frac{1}{1 + x + y}\, dA,\ R = [1, 3] \times [1, 2]$

Volumes by Iterated Integrals

1. Find the volume of the solid that lies under the plane

$$4x + 6y - 12z + 15 = 0$$

and above the rectangle $[-1, 2] \times [-1, 1]$

2. Find the volume of the solid lying under the elliptic paraboloid

$$x^2/4 + y^2/9 + z = 1$$

and above the rectangle $[-1, 1] \times [-2, 2]$

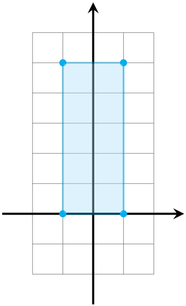
Average Values

The average value of $y = f(x)$ on $[a, b]$ is

$$f_{\mathrm{av}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The average value of $z = f(x, y)$ on a rectangle R with area $A(R)$ is

$$f_{\mathrm{av}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$$



Find the average value of $f(x, y) = x^2 y$ over a rectangle with vertices $(-1, 0)$, $(-1, 5)$, $(1, 5)$, and $(1, 0)$.