§ 15.2 Double Integrals: Gen'l Regions.

ex. $\iint_R xy \, dA$

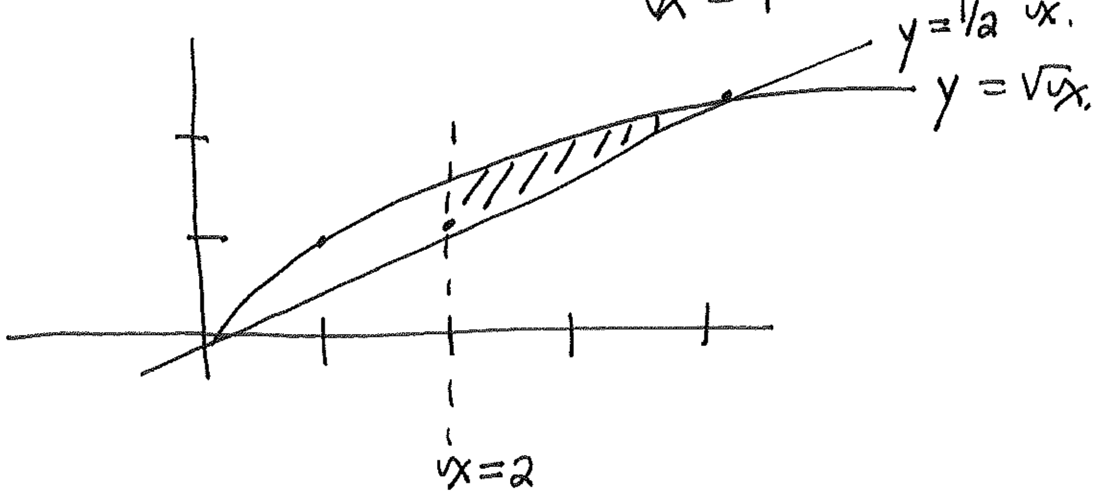
where R is enclosed between

$y = \frac{1}{2}x$

$y = \sqrt{x}$

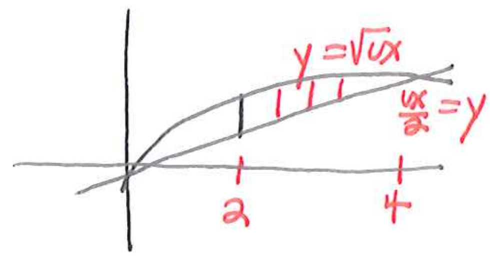
$x = 2$

$x = 4$



Solution #1

$$\int_2^4 \int_{\frac{x}{2}}^{\sqrt{x}} xy \; dy \; dx$$



Fix x.
y ranges from $\frac{x}{2}$ to $\sqrt{x}$

$$= \int_2^4 x \cdot \frac{y^2}{2} \Big]_{x/2}^{\sqrt{x}} \; dx$$

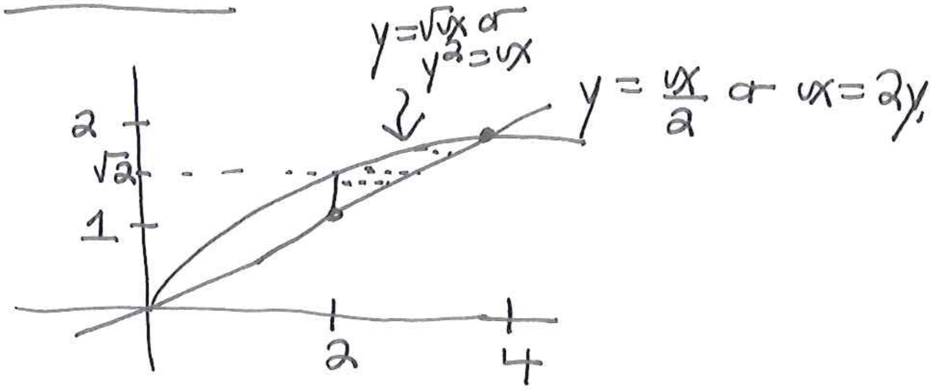
$$= \int_2^4 x \cdot \left(\frac{x}{2} - \frac{x^2}{8} \right) dx$$

$$= \frac{x^3}{6} - \frac{x^4}{32} \Big]_2^4 = \frac{64}{6} - \frac{16 \cdot 16}{32} - \left(\frac{8}{6} - \frac{16}{32} \right)$$

$$= \frac{56}{6} - \frac{16 \cdot 15}{32} = \frac{56}{6} - \frac{15}{2}$$

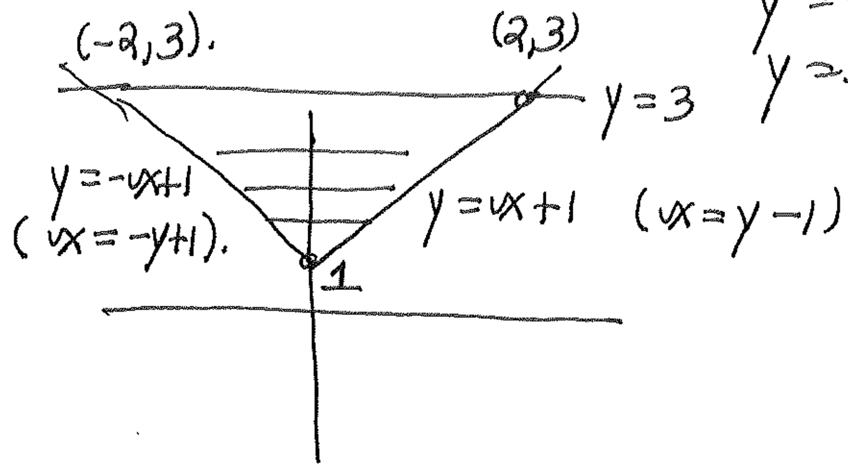
$$= \frac{56}{6} - \frac{45}{6} = \frac{11}{6}$$

Sol'n #2 Other direction.

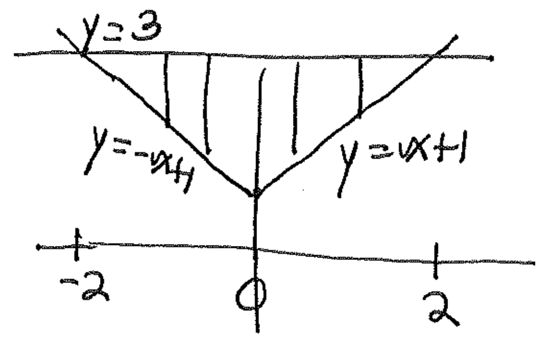


$$\int_1^{\sqrt{2}} \int_2^{2y} xy \; dx \, dy + \int_{\sqrt{2}}^{2} \int_{y^2}^{2y} xy \; dx \, dy \; .$$

ex. $\displaystyle\iint_R (2x-y^2)\,dA$ where R is enclosed by
$y=-x+1$
$y=x+1$
$y=3$.



$$\int_1^3 \int_{-y+1}^{y-1} (2x-y^2)\,dx\;dy.$$



Other direction

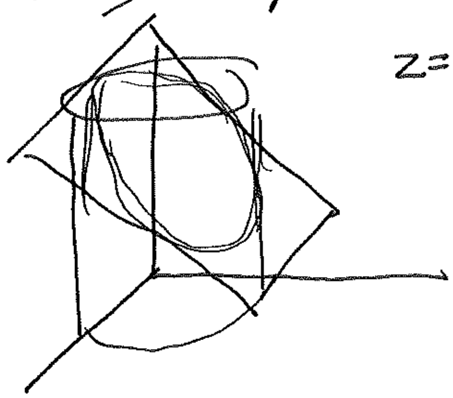
$$\int_{-2}^{0} \int_{-x+1}^{3} (2x-y^2)\,dy\,dx$$

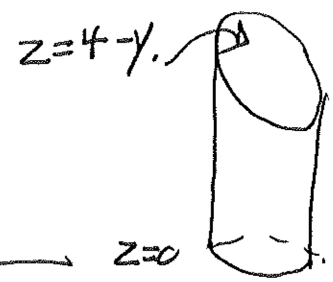
$$+\int_0^2 \int_{x+1}^{3} (2x-y^2)\,dy\,dx.$$

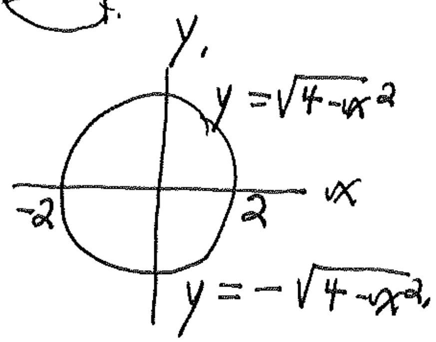
ex. Find the volume of the solid bounded by the cylinder

$$x^2+y^2=4$$

+ the planes $y+z=4$ + $z=0$.







$$V=\iint_R 4-y \; dy.$$

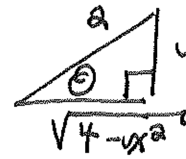
$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{y=+\sqrt{4-x^2}} (4-y)\,dy\,dx.$$

$$=\int_{-2}^{2} 4y-\frac{y^2}{2}\Big]_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \; dx$$

$$=\int_{-2}^{2} 8\sqrt{4-x^2}\; dx.$$

Let $x = 2\sin\theta \quad \Rightarrow \sin\theta = x/2$

$dx = 2\cos\theta \, d\theta$



$-2 \le x \le 2$

$-1 \le \frac{x}{2} \le 1$

$-1 \le \sin\theta \le 1$

$-\pi/2 \le \theta \le \pi/2$

$$= \int_{-\pi/2}^{\pi/2} 8 \cdot \sqrt{4 - 4\sin^2\theta} \cdot 2\cos\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 8 \cdot \sqrt{4\cos^2\theta} \cdot 2\cos\theta \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 32 \cdot \cos^2\theta \, d\theta$$

Recall $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$.

$$= 32 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2\theta}{2} \, d\theta$$

$$= 32 \cdot \left(\frac{1}{2}\theta + \underbrace{\frac{\sin 2\theta}{4}}_{0.} \right]_{-\pi/2}^{\pi/2.}$$

$$= 32 \cdot \frac{1}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= 16 \cdot \pi$$

Properties

R any region.

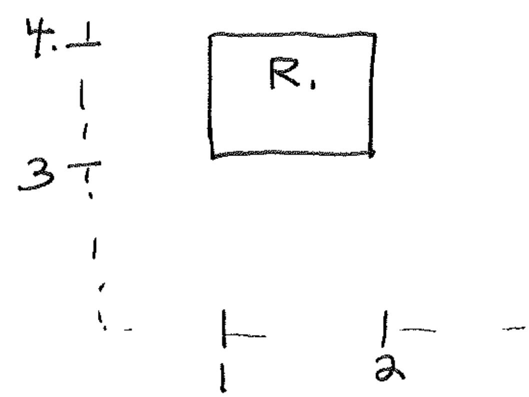
① $$\iint_R f(x,y) + g(x,y)\, dA = \iint_R f\, dA + \iint_R g\, dA.$$

② $$\iint_R c \cdot f(x,y)\, dA = c \cdot \iint_R f(x,y)\, dA.$$

③ If $g(x,y) \le f(x,y)$ for all $(x,y) \in R$
then

$$\iint_R g(x,y) \le \iint_R f(x,y).$$

ex. $$\iint_R xy\, dA \qquad R = [1,2] \times [3,4].$$



$3 \le xy \le 8$ on R

So

$$\iint_R 3\, dA \le \iint_R xy\, dA \le \iint_R 8\, dA$$

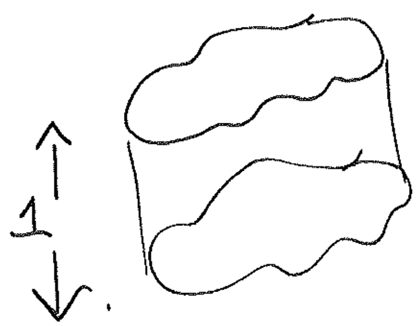
$\iint_R 3\, dA = 3 \cdot 1 = 3.$

$\iint_R 8\, dA = 8 \cdot \text{area}(R) = 8 \cdot 1 = 8$

④. Average value of a function f on a region R is

$$\frac{1}{\text{Area } R} \iint_R f \, dA$$

⑤ $\iint_R 1 \cdot dA =$ Area of region R

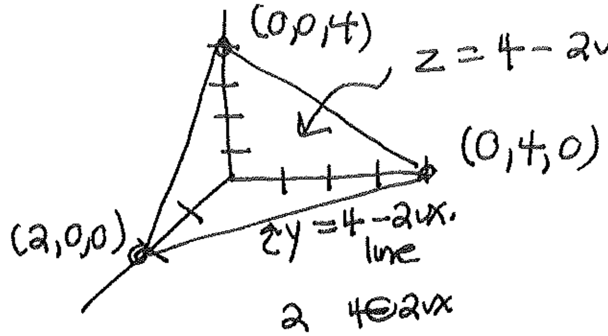


geometrically = volume

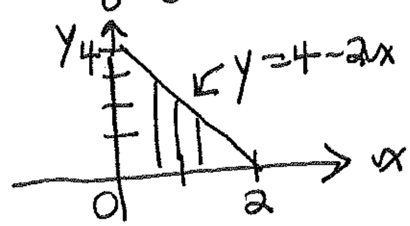
⑥ If $R = R_1 \cup R_2$, then

$$\iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA$$

ex. Find volume of tetrahedron enclosed by coord. planes + plane, $2x+y+z=4$.



$$\text{Volume} = \int_0^2 \int_0^{4-2x} 4-2x-y. \quad dy.dx$$



$$= \int_0^2 \left. 4y - 2xy - \frac{y^2}{2} \right]_0^{4-2x} dx$$

$$= \int_0^2 4(4-2x) - 2x(4-2x) - \frac{(4-2x)^2}{2}\, dx$$

Miscopied limits of integration here — will affect final answer.

$$= \int_0^4 16 - 16x + 4x^2 - \frac{(4-2x)^2}{2}\, dx$$

$$= 16x - \frac{16x^2}{2} + 4\frac{x^3}{3} + \left. \frac{(4-2x)^3}{12} \right]_0^4$$

$$= 16(4) - 16(8) + \frac{(16)^2}{3} + \frac{4^3}{12}$$

Should be a minus here

$$= 16(4)\left(1 - 2 + \frac{4}{3} + \frac{1}{12}\right)$$

$$-1 + \frac{16}{12} + \frac{1}{12} = \frac{5}{12}$$

The final answer really is 16/3.

$$= 16 \cdot 4 \cdot \frac{5}{12\,3} = \boxed{\frac{80}{3}}$$

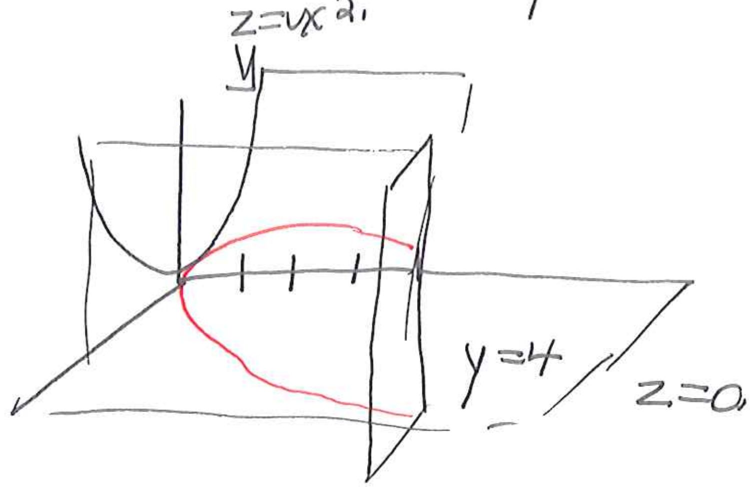
Algo

$$\text{Vol (tetra)} = \frac{1}{3}\left(\begin{matrix}\text{Area}\\ \text{base}\end{matrix}\right) \cdot \text{height}$$

$$= \frac{1}{3}\left(\frac{1}{2} \cdot 2 \cdot 4\right) \cdot 4$$

ex. Vol. enclosed by $z = x^2$, $y = x^2$ } cylinders,

+ planes $z = 0$, $y = 4$

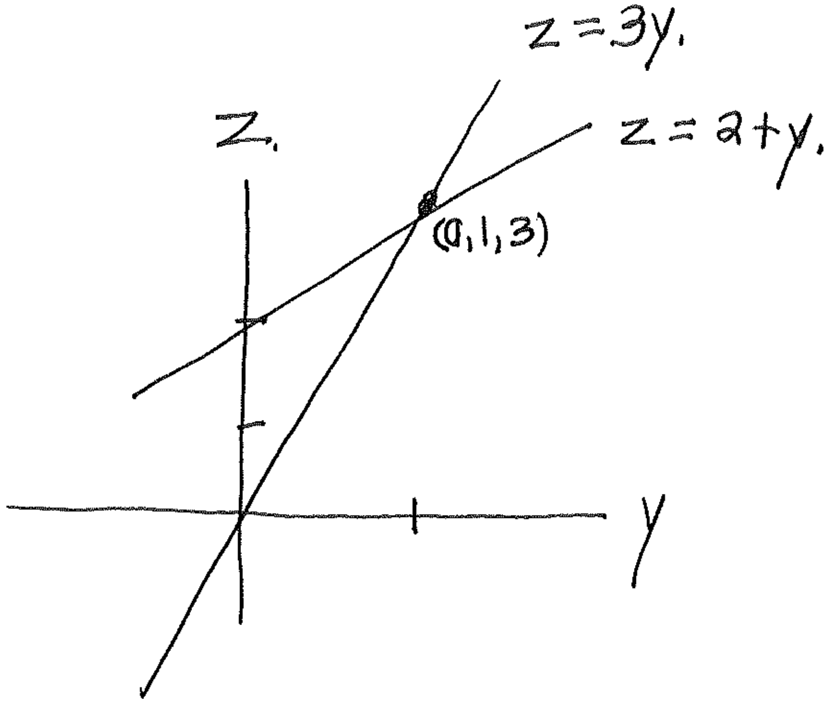


Volume enclosed by

$y = x^2$

& $z = 3y$

$z = 2 + y.$



$\Big\}\ 0 \le y \le 1.$