Math 213 - Double Integrals Over General Regions

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Homework

- Re-read section 15.2
- Begin work on problems 1-25 (odd), 35, 37, 45-55 (odd), 61, 65 from 15.2
- Read section 15.3 for Monday, October 22
- Begin Webwork C1

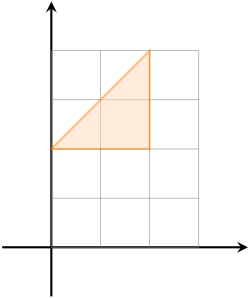
Unit III: Multiple Integrals

Lecture 24	Double Integrals over Rectangles
Lecture 25	Double Integrals over General Regions
Lecture 26	Double Integrals in Polar Coodinates
Lecture 27	Applications of Double Integrals
Lecture 28	Surface Area
Lecture 29	Triple Integrals
Lecture 30	Triple Integrals in Cylindrical Coordinates
Lecture 31	Triple Integrals in Spherical Coordinates
Lecture 32	Change of Variable in Multiple Integrals, Part I
Lecture 33	Change of Variable in Multiple Integrals, Part II
Lecture 34	Exam III Review

Goals of the Day

- Learn how to set up iterated integrals for double integrals over plane regions of Type I and Type II
- Learn properties of double integrals

Integrals over General Regions



Example Find $\iint_D (2x+y)\,dA$ if

$$D = \{(x,y) : 1 \leq y \leq 2,\ y-1 \leq x \leq 1\}$$

Integrals over General Regions

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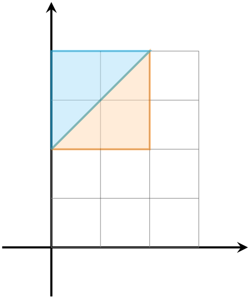
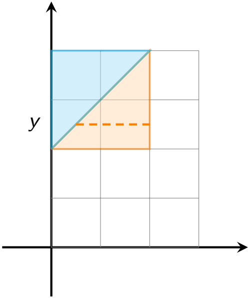


We can view this as an integral over the rectangle $[0,1] \times [1,2]$ if we set

$$f(x,y) = \begin{cases} 2x+y & x \ge y-1 \text{ (orange)} \\ 0 & x \le y-1 \text{ (blue)} \end{cases}$$

Integrals over General Regions



Example Find $\iint_D (2x + y)\, dA$ if

$$D = \{(x, y) : 1 \leq y \leq 2,\ y - 1 \leq x \leq 1\}$$

We can view this as an integral over the rectangle $[0, 1] \times [1, 2]$ if we set

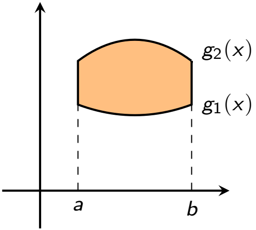
$$f(x, y) = \begin{cases} 2x + y & x \geq y - 1 \text{ (orange)} \\ 0 & x \leq y - 1 \text{ (blue)} \end{cases}$$

Then

$$\int_1^2 \int_0^1 f(x, y)\, dx\, dy = \int_1^2 \int_{y-1}^1 (2x + y)\, dx\, dy$$

This can be computed as an iterated integral–do it!

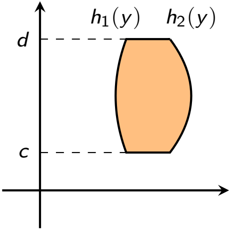
Integrals Over General Regions



We'll see how to compute $\iint_R f(x, y)\, dA$ if R is one of the following kinds of regions:

Type I: R lies between the graphs of two continuous functions of x

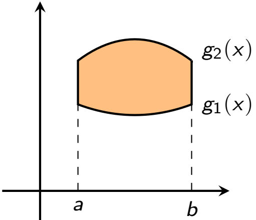
$$D = \{(x, y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$



Type II: R lies between the graphs of two continuous functions of y

$$D = \{(x, y) : c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\}$$

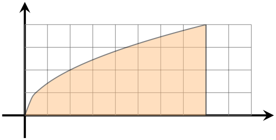
Double Integrals Over Type I Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

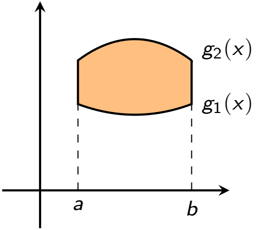
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



Example: Find $\iint_D \frac{y}{x^2+1}\,dA$ if

$$D = \{(x,y) : 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$$

Double Integrals Over Type I Regions



To compute $\iint_D f(x, y)\, dA$ if

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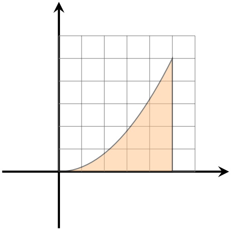
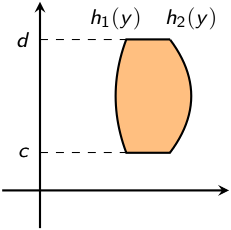


$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

Example: Find $\iint_R x \cos y\, dA$ if D is the region bounded by $y = 0$, $y = x^2$, and $x = 1$

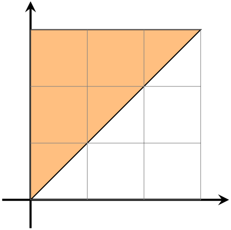
Double Integrals Over Type II Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

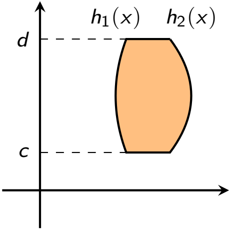
$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$



Example: Find $\iint_D e^{-y^2}\,dA$ if

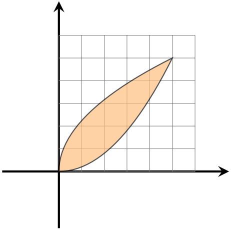
$$D = \{(x,y) : 0 \le y \le 3,\ 0 \le x \le y\}$$

Double Integrals over Type II Regions



To compute $\iint_D f(x,y)\,dA$ if

$$D = \{(x,y) : c \le y \le d, h_1(y) \le x \le h_2(y)\}$$



$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Example: Find the volume under the plane $3x + 3y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

Type I or Type II?

Find the best way to compute each of the following volumes.

1. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$
2. The volume enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

Properties of Double Integrals, Part I

1. (linearity) $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$
2. (linearity) $\iint_D cf(x,y)\, dA = c \iint f(x,y)\, dA$
3. (order) If $f(x,y) \geq g(x,y)$ for all $(x,y) \in D$, then
$$\iint_D f(x,y)\, dA \geq \iint_D g(x,y)\, dA$$
4. $\iint_D 1\, dA = A(D)$ where $A(D)$ is the area of the domain D

Find the volume of the the solid by subtracting two volumes:

The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$ and $2x + 2y - z + 10 = 0$

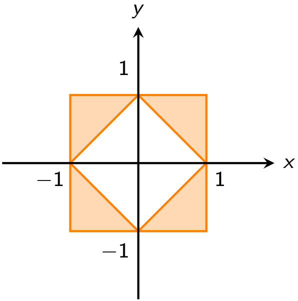
Properties of Double Integrals, II

1. (*additivity) If $D = D_1 \cup D_2$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

2. (order) If $m \le f(x,y) \le M$ then

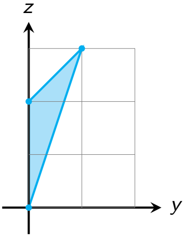
$$mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$$



Express $\iint_D xy\,dA$ as a union of type I and type II integrals if D is as shown

Volumes of Solids - Subtracting Two Volumes

We'll use the GeoGebra package at www.geogebra.org/3d to figure out what's going on here!



Find the volume of the solid enclosed by the parabolic cylinder

$$y = x^2$$

and the planes

$$z = 3y$$

and

$$z = 2 + y$$

Hint: It helps to consider the surface as two graphs $x = \pm\sqrt{y}$ over the yz plane!