Math 213 - Double Integrals in Polar Coordinates

Peter A. Perry

University of Kentucky

October 22, 2018

Homework

- Re-read section 15.3
- Begin work on 1-4, 5-31 (odd), 35, 37 from 15.3
- Read section 15.4 for Wednesday, October 24
- Continue working on Webwork C1

Unit III: Multiple Integrals

Lecture 24	Double Integrals over Rectangles
Lecture 25	Double Integrals over General Regions
Lecture 26	Double Integrals in Polar Coodinates
Lecture 27	Applications of Double Integrals
Lecture 28	Surface Area
Lecture 29	Triple Integrals
Lecture 30	Triple Integrals in Cylindrical Coordinates
Lecture 31	Triple Integrals in Spherical Coordinates
Lecture 32	Change of Variable in Multiple Integrals, Part I
Lecture 33	Change of Variable in Multiple Integrals, Part II
Lecture 34	Exam III Review

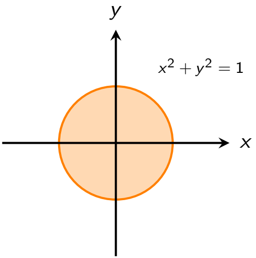
Goals of the Day

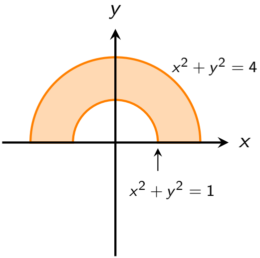
- Review Polar Coordinates, introduce Polar Rectangles
- Learn how to compute double integrals over polar rectangles
- Learn how to compute double integrals over polar regions
- Learn to compute volumes using polar integrals

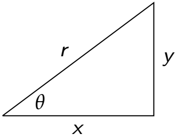
Reality Check

	Calculus I	**Calculus III**
Riemann sum	$\sum_{i=1}^{n} f(x_i^*)\Delta x$	$\sum_{i,j=1}^{n} f(x_i^*, y_j^*)\Delta A$
Riemann Integral	$\int_a^b f(x)\, dx$	$\iint_D f(x,y)\, dA$
Way of computing	$F(b) - F(a)$	Iterated Integral
Interpretation	Area under a curve	Volume under a surface

Review of Polar Coordinates







Recall that

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

and

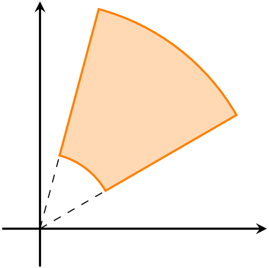
$$x = r\cos\theta, \quad y = r\sin\theta.$$

How would you describe the regions at left in polar coordinates?

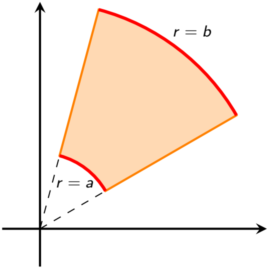
Polar Rectangles

A *polar rectangle* is a region

$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$



Polar Rectangles



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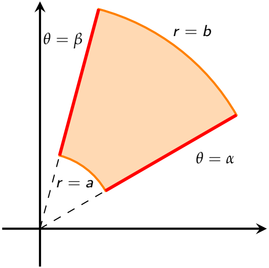
Polar Rectangles



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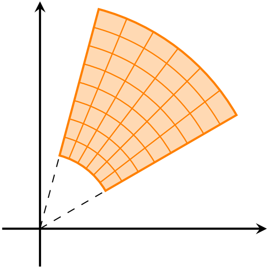
Polar Rectangles



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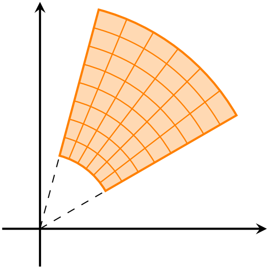
$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$

Like an ordinary rectangle a polar rectangle can be divided into *subrectangles*

Polar Rectangles



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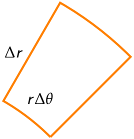
$$R = \{(r, \theta) : a \leq r \leq b, \quad \alpha \leq \theta \leq \beta\}.$$

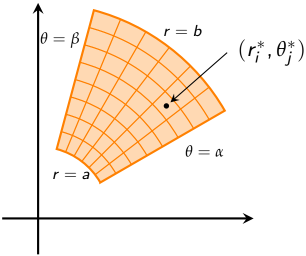
Like an ordinary rectangle a polar rectangle can be divided into *subrectangles*

A small polar rectangle has area

$$\Delta A \simeq r\,\Delta r\,\Delta\theta$$



Integrals Over Polar Rectangles



The double integral $\iint_R f(x,y)\,dA$ is a limit of Riemann sums:

$$\sum_{i,j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_j\, \Delta r\, \Delta\theta$$

Rectangle R_{ij} is given by

$$R_{ij} = \{(r,\theta) : r_{i-1} \le r \le r_i, \theta_{j-1} \le \theta \le \theta_j\}$$

$$r_i = a + i\Delta r \quad , \theta_j = \alpha + j\Delta\theta$$

where

$$\Delta r = \frac{b-a}{n}, \quad \Delta\theta = \frac{\beta - \alpha}{n}$$

In the limit this leads to an iterated integral

$$\int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Integrals Over Polar Rectangles

Double Integral In Polar Coordinates The integral of a continuous function $f(x, y)$ over a polar rectange R given by $a \leq r \leq b$, $\alpha \leq r \leq \beta$, is

$$\iint_R f(x, y)\, dA = \int_{\alpha}^{\beta} \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

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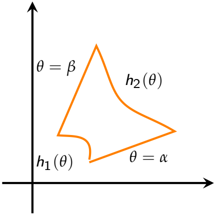
1. Find $\iint_R (2x - y)\, dA$ if R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
2. Find $\iint_R e^{-x^2 - y^2}\, dA$ if D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

Integrals over Polar Regions



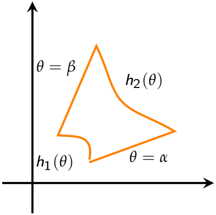
If f is continuous over a polar region of the form

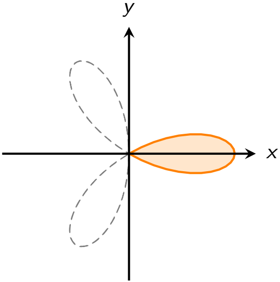
$$D = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Integrals over Polar Regions





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$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

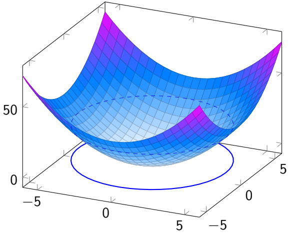
then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Find the area of one loop of the rose

$$r = \cos 3\theta$$

Volumes of Solids



Find the volume under the paraboloid

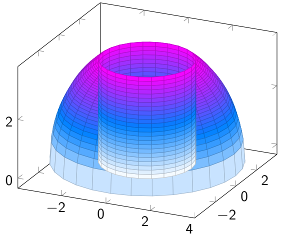
$$z = x^2 + y^2$$

and above the disc

$$x^2 + y^2 < 25$$

1. Describe the disc in polar coordinates
2. Transform $f(x, y)$ to polar coordinates

Volumes of Solids



Find the volume inside the sphere

$$x^2 + y^z + z^2 = 16$$

and outside the cylinder

$$x^2 + y^2 = 4$$