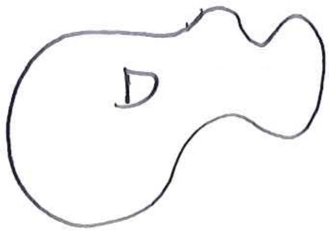
bring ruler & cardboard cutouts.

§ 15.~~5~~4. Double integrals: Applications.



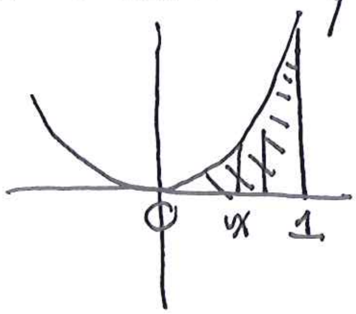
lamina.
"a thin layer of some material."
(rock / tissue / ...).

total mass of lamina $= \iint_D \underbrace{\rho(x,y)}_{\text{density function (mass per unit area @ point } (x,y))} \, dA.$

Other densities:

$\sigma(x,y) =$ charge density.
(units of charge per unit area).

ex. lamina, $y = x^2$ w/ density function $\rho(x,y) = y$.

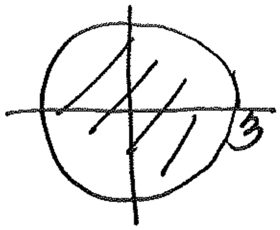


$$\text{mass} = \int_0^1 \int_0^{x^2} y \, dy \, dx$$

$$= \int_0^1 \left. \frac{y^2}{2} \right]_0^{x^2} dx$$

$$= \int_0^1 \frac{x^4}{2} \, dx = \left. \frac{x^5}{10} \right]_0^1 = 1/10$$

ex. Charge distr. over circular region D



charge density is $\sigma(x,y) = \cancel{\cos}\, x$. coulombs per sq. m.

Find total charge.

$$\int_0^{2\pi}\int_0^{3} r\cos\theta \; r\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big]_0^3 \cos\theta\, d\theta$$

$$= \int_0^{2\pi} 9\cos\theta\, d\theta = -9\sin\theta\Big]_0^{2\pi} = -9(\sin 2\pi - \sin 0)$$

$$= -9(0-0)$$

$$= 0. \text{ coulombs}$$

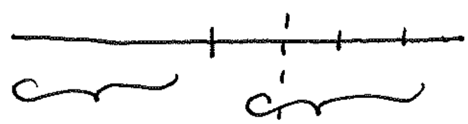
(Why?).

Center of mass

← uniform density.

fulcrum.

$\leftarrow \frac{\Delta x}{2} \rightarrow \leftarrow \frac{\Delta x}{2} \rightarrow$



← Not uniform density.

$\rho(\Delta x) = 1$ $\qquad$ $\rho(\Delta x) = 2$

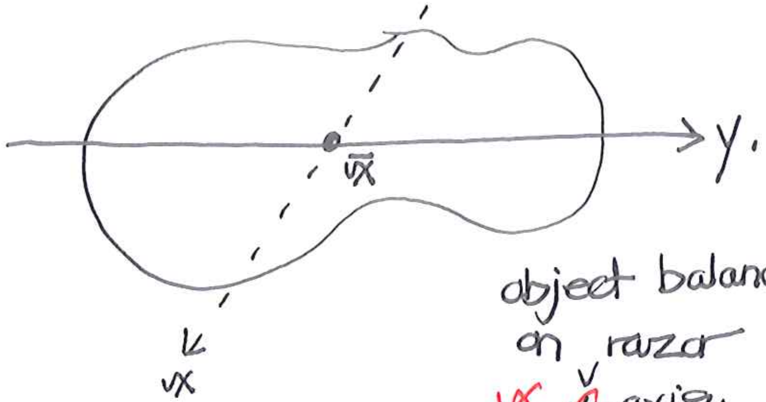
new center of mass

$\leftarrow \frac{5}{8}\Delta x \rightarrow \quad \leftarrow \frac{3}{8}\Delta x \rightarrow$

(mass is $1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{8} = \frac{3}{4}$) $\qquad$ (mass is $\frac{3}{8} \cdot 2 = \frac{6}{8}$).

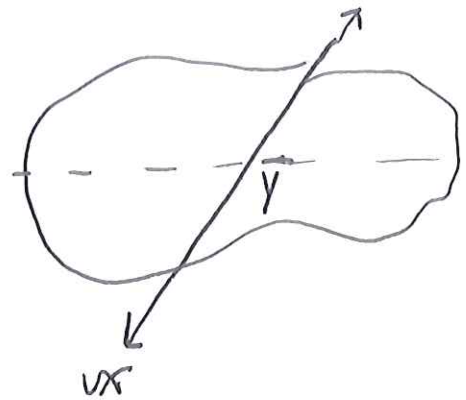
Fancy defn:
center of mass is the point where the weighted relative position of the mass sums to zero.

Can replace object w/ a particle of that mass at the center of mass to study the effect of forces on a body/object.



object balances on razor x ~~y~~-axis.

when. $y = \bar{x}$



object balances on ~~x~~ y-axis razor

when $x = \bar{y}$

Moment about x-axis.

$$M_x = \iint_D y \cdot \rho(x,y)\, dA$$

Moment about y-axis

$$M_y = \iint_D x \cdot \rho(x,y)\, dA$$

Center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{m} \iint_D \cancel{y}\, x \cdot \rho(x,y)\, dA$$

$$= \frac{M_{\cancel{x}\, y}}{m}$$

$$\bar{y} = \frac{1}{m} \iint_D \cancel{x}\, y\, \rho(x,y)\, dA$$

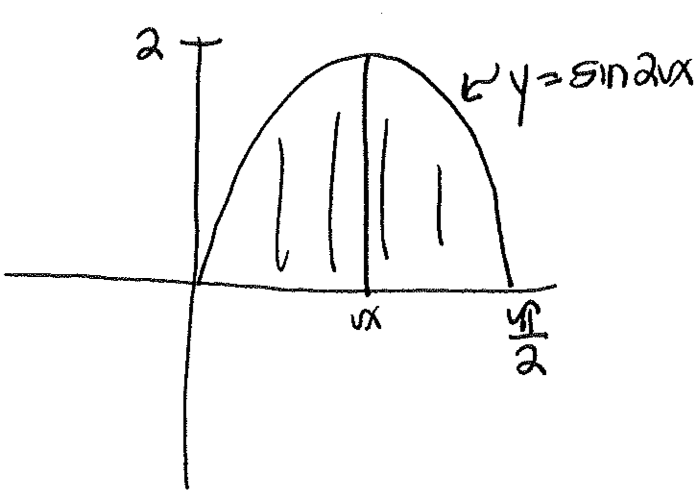
$$= \frac{M_{\cancel{y}\, x}}{m}.$$

ex. Find center of mass. for region bnded by

$$y = 2\sin(2x), \quad y=0, \quad 0 \le x \le \pi/2.$$

assuming uniform density fn. $\rho(x,y) = 1$.

$$\text{mass} = m = \int_0^{\pi/2}\int_0^{2\sin 2x} 1.\ dy\ dx$$



$$= \int_0^{\pi/2} 2\sin 2x\, dx$$

$$= -\cancel{\tfrac{2}{2}} \cos 2x \Big]_0^{\pi/2}$$

$$= -(\cos\pi - \cos 0)$$

$$= -((-1) - 1) = 2.$$

$$M_x = \int_0^{\pi/2}\int_0^{2\sin 2x} y \cdot 1 \cdot dy\, dx = \int_0^{\pi/2} \frac{y^2}{2}\Big]_0^{2\sin 2x} dx$$

$$= \int_0^{\pi/2} 2\sin^2 2x\, dx = 2\int_0^{\pi/2} \left(\frac{1-\cos 4x}{2}\right) dx$$

$$= \int_0^{\pi/2} 1 - \cos 4x\, dx = x - \frac{1}{4}\sin 4x\Big]_0^{\pi/2} \quad \pi/2$$

$$= \frac{\pi}{2} - \frac{1}{4}\sin 2\pi - \left(0 - \frac{1}{4}\cdot\sin 0\right)$$

$$M_y = \int_0^{\pi/2} \int_0^{2\sin 2x} x \cdot 1 \cdot dy \, dx$$

$$= \int_0^{\pi/2} 2x \sin 2x \, dx$$

$\int$ by parts

let $u = x$ $\qquad$ $dv = 2\sin 2x \, dx$

$du = dx$ $\qquad$ $v = -\cos 2x$

$$u \cdot v - \int v \, du$$

$$= x(-\cos 2x)\Big]_0^{\pi/2} - \int_0^{\pi/2} -\cos 2x \, dx$$

$$= -\frac{\pi}{2}\cos \pi - 0 \cdot \cos 0 + \frac{1}{2}\sin 2x\Big]_0^{\pi/2}$$

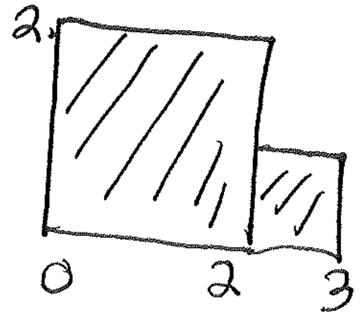
$$= -\frac{\pi}{2}(-1) + \frac{1}{2}(\underbrace{\sin \pi - \sin 0}_{0})$$

$$= \pi/2.$$

Thus $\bar{x} = \frac{1}{2} \cdot \frac{\pi}{2} = \pi/4$, $\quad \bar{y} = \frac{1}{2} \cdot \frac{\pi}{2} = \pi/4.$

(Try w/ cardboard model).

ex



$\rho(x,y)=1$

$m = 5. = \text{area}.$

$$I_x = \int_0^2 \int_0^2 y \, dy \, dx + \int_2^3 \int_0^1 y \, dy \, dx$$

$$= \int_0^2 \left.\frac{y^2}{2}\right]_0^2 dx + \int_2^3 \left.\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^2 2 \, dx + \int_2^3 \frac{1}{2} \, dx$$

$$= \left.2x\right]_0^2 + \left.\frac{1}{2}x\right]_2^3 = 4 + \frac{1}{2} = \frac{9}{2}.$$

$$\bar{x} = \frac{1}{5} \cdot \frac{9}{2} = \frac{9}{10}. = .9.$$

I_y

$$I_y = \int_0^2 \int_0^2 x \cdot dy\, dx + \int_2^3 \int_0^1 x\, dy\, dx$$

$$= \int_0^2 2x\, dx + \int_2^3 x\, dx$$

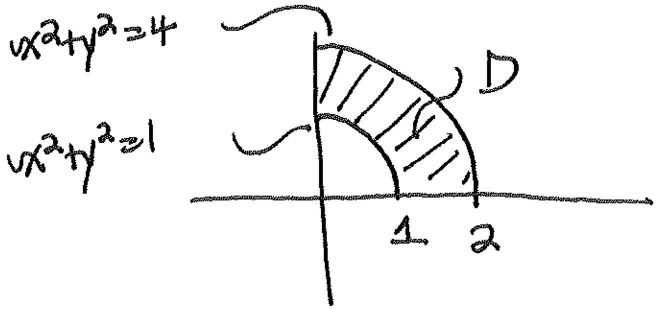
$$= x^2 \Big]_0^2 + \frac{x^2}{2} \Big]_2^3$$

$$= 4 + \left(\frac{9}{2} - 2\right)$$

$$= 2 + \frac{9}{2} = \frac{13}{2}.$$

$$\bar{y} = \frac{1}{m} \cdot I_y = \frac{1}{5} \cdot \frac{13}{2} = \frac{13}{10} = 1.3$$

ex. Find center of mass of.



$\rho(x,y) = \sqrt{x^2+y^2}$,

mass

$$m = \iint_D \underbrace{\sqrt{x^2+y^2}}_{r} \, dA.$$

Use polar coords!

$$m = \int_0^{\pi/2}\int_1^2 r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big]_1^2 \, d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3} - \frac{1}{3} \, d\theta = \int_0^{\pi/2} \frac{7}{3} \, d\theta$$

$$= \frac{7}{3} \cdot \frac{\pi}{2} = \frac{7\pi}{6}$$

Moment about x-axis.

$$I_x = \iint_D y \cdot \sqrt{x^2+y^2}\, dA$$

$$= \int_0^{\pi/2}\int_1^2 r\sin\theta \cdot r \cdot \, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \cdot \left.\frac{r^4}{4}\right]_1^2 d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left(4 - \frac{1}{4}\right) d\theta = \frac{15}{4}\int_0^{\pi/2} \sin\theta\, d\theta$$

$$= \frac{15}{4} \cdot \left.(-\cos\theta)\right]_0^{\pi/2}$$

$$= \frac{15}{4}\left(-0 - (-1)\right) = \frac{15}{4}$$

Moment about y-axis.

$$I_y = \iint_D x \cdot \sqrt{x^2+y^2} \, dA$$

$$= \int_0^{\pi/2} \int_1^2 r\cos\theta \cdot r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta \cdot \frac{r^4}{4}\Big]_1^2 \, d\theta$$

$$= \frac{15}{4} \int_0^{\pi/2} \cos\theta \, d\theta$$

$$= \frac{15}{4} \left(\sin\theta \right)\Big]_0^{\pi/2} = \frac{15}{4}\left(\sin \pi/2 - \sin 0\right)$$

$$= \frac{15}{4}(1-0) = \frac{15}{4}.$$

So

$$\bar{x} = \frac{1}{m} \cdot I_{\cancel{x}y} = \frac{6}{7\pi} \cdot \frac{15}{4} = \frac{45}{14\pi} < 1$$

$$\bar{y} = \frac{1}{m} \cdot I_{\cancel{y}x} = \frac{6}{7\pi} \cdot \frac{15}{4} \quad '' \quad < 1.$$

$\Big\}$ Not on D! Why? see (balancing bird toy).

Moment of inertia. — think ~~ice~~ skater w/ arms extended vs. tucked in.

mr vs mr^2.

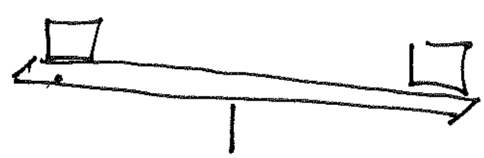
$$I_x = \iint_D y^2 \cdot \rho(x,y)\, dA.$$

(about x-axis)

$$I_y = \iint x^2 \rho(x,y)\, dA$$

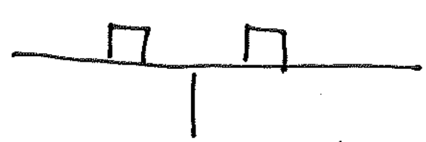
(about y-axis).

mass	moment of inertia.
linear motion.	rotational motion.



spin

vs.



spin (easier).

$$\bar{x} = \frac{1}{m} \iint x \cdot \rho(x,y)\, dA$$

$$\bar{y} = \frac{1}{m} \iint y \cdot \rho(x,y)\, dA.$$

ex. Moment of inertia.

about x-axis.

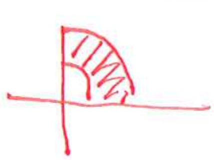
$$I_x = \iint_D y^2 \rho(x,y)\, dA$$

about y-axis

$$I_y = \iint_D x^2 \rho\, dA.$$

about origin.

$$\iint_D (x^2+y^2)\, \rho\, dA$$

ex. Use  + polar coords.