Math 213 - Surface Area

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October 29, 2018

Homework

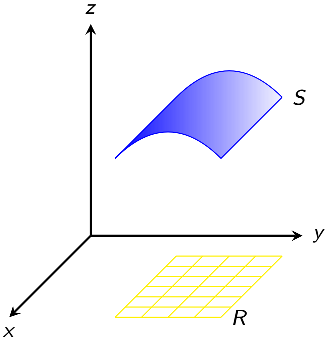
- Re-read section 15.5
- Begin work on problems 1-11 (odd) in section 15.5
- Read section 15.6 for Wednesday October 31
- Work on Webwork C2

Unit III: Multiple Integrals

Goals of the Day

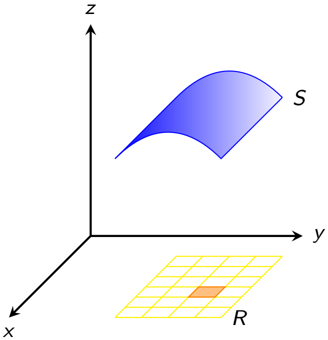
- Understand surface area as a limit of Riemann sums
- Understand how to compute surface area as a double integral

Surface Area



Suppose S is a surface in $\mathbb{R}^3$ over a rectangle R.

Surface Area



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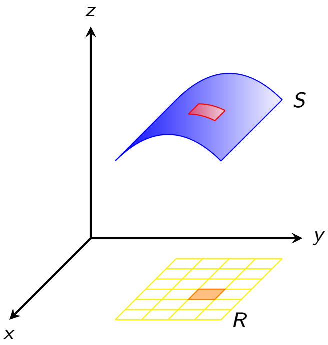
Over each rectangle R_{ij} in the xy plane

Surface Area



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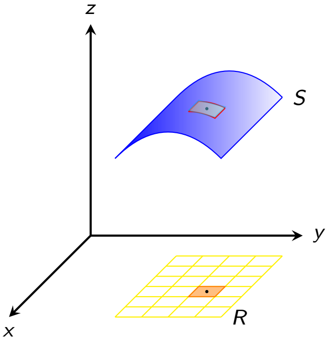
Surface Area



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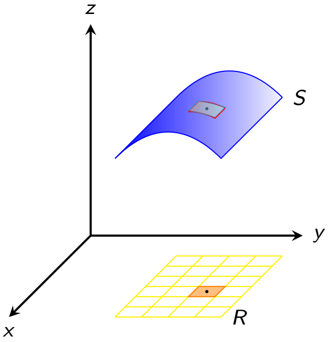
We'll estimate the area of this piece of surface by using the tangent plane to S

Surface Area



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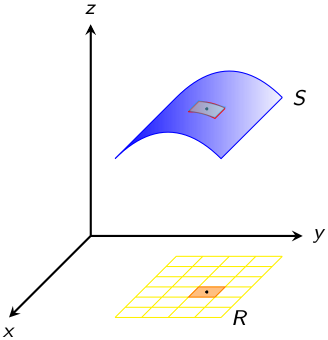
Take the linear approximation from the midpoint:

$$L(x, y) = f(x_i^*, y_i^*) + f_x(x_i^*, y_i^*)(x - x_i^*) + f_y(x_i^*, y_i^*)(y - y_i^*)$$

Surface Area



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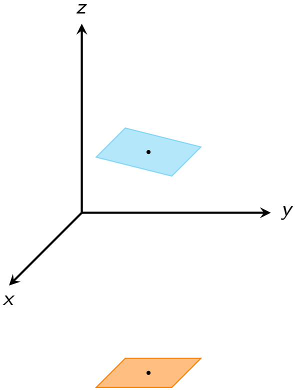
Find the vectors **a** and that span the cyan rectangle, and compute the area $A = |\mathbf{a} \times \mathbf{b}|$

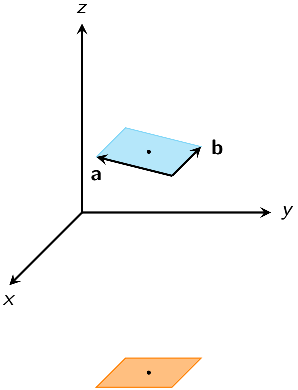
Surface Area



Let

$$f_x(x_i^*, y_i^*) = a, \quad f_y(x_i^*, y_i^*) = b$$

Surface Area



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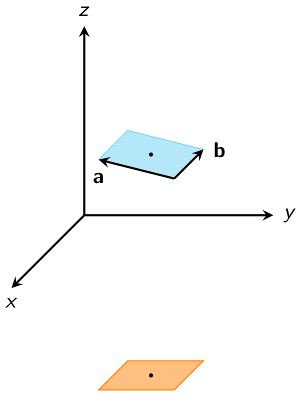
The parallelogram is spanned by the vectors

$$\begin{aligned} \mathbf{a} &= (\Delta x, 0, a\Delta x) \\ \mathbf{b} &= (0, \Delta y, b\Delta y) \\ \mathbf{a} \times \mathbf{b} &= (-a\mathbf{i} - b\mathbf{j} + \mathbf{k})\, \Delta x\, \Delta y \end{aligned}$$

Surface Area



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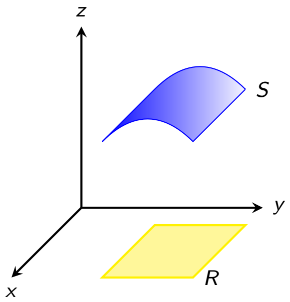
The parallelogram has area

$$\begin{aligned} \Delta A &= |\mathbf{a} \times \mathbf{b}| \\ &= \sqrt{1 + a^2 + b^2}\, \Delta x \Delta y \end{aligned}$$

Surface Area



If $R = [a,b] \times [c,d]$, then, the surface area of S is given by

$$\int_a^b \int_c^d \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dx\, dy$$

For a general region R, the area of a surface over R is

$$\iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

. . . so any method of evaluating this double integral (type I, type II, polar) is fair game!

Surface Area

For a general region R, the area of a surface over R is

$$\iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Find the surface area of ...

1. The part of the plane $5x + 3y - z + 6 = 0$ above the rectangle $[1, 4] \times [2, 6]$
2. The part of the surface $2y + 4z = x^2 = 5$ above the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 4)$
3. The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$
4. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$