# Math 213 - Surface Area 

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## Homework

- Re-read section 15.5
- Begin work on problems 1-11 (odd) in section 15.5
- Read section 15.6 for Wednesday October 31
- Work on Webwork C2


## Unit III: Multiple Integrals

Lecture 24 Double Integrals over Rectangles
Lecture 25 Double Integrals over General Regions
Lecture 26 Double Integrals in Polar Coodinates
Lecture 27 Applications of Double Integrals
Lecture 28 Surface Area

Lecture 29 Triple Integrals
Lecture 30 Triple Integrals in Cylindrical Coordinates
Lecture 31 Triple Integrals in Spherical Coordinates
Lecture 32 Change of Variable in Multiple Integrals, Part I
Lecture 33 Change of Variable in Multiple Integrals, Part II
Lecture 34 Exam III Review

## Goals of the Day

- Understand surface area as a limit of Riemann sums
- Understand how to compute surface area as a double integral


## Surface Area



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Take the linear approximation from the midpoint:

$$
L(x, y)=f\left(x_{i}^{*}, y_{i}^{*}\right)+f_{x}\left(x_{i}^{*}, y_{i}^{*}\right)\left(x-x_{i}^{*}\right)+f_{y}\left(x_{i}^{*}, y_{i}^{*}\right)\left(y-y_{i}^{*}\right)
$$

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$$

Find the vectors a and that span the cyan rectangle, and compute the area $A=|\mathbf{a} \times \mathbf{b}|$

## Surface Area



## Surface Area

Let

$$
f_{x}\left(x_{i}^{*}, y_{i}^{*}\right)=a, \quad f_{y}\left(x_{i}^{*}, y_{i}^{*}\right)=b
$$

The parallelogram is spanned by the vectors

$$
\begin{aligned}
\mathbf{a} & =(\Delta x, 0, a \Delta x) \\
\mathbf{b} & =(0, \Delta y, b \Delta y) \\
\mathbf{a} \times \mathbf{b} & =(-a \mathbf{i}-b \mathbf{j}+\mathbf{k}) \Delta x \Delta y
\end{aligned}
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\end{aligned}
$$

The parallelogram has area

$$
\begin{aligned}
\Delta A & =|\mathbf{a} \times \mathbf{b}| \\
& =\sqrt{1+a^{2}+b^{2}} \Delta x \Delta y
\end{aligned}
$$

## Surface Area



If $R=[a, b] \times[c, d]$, then, the surface area of $S$ is given by

$$
\int_{a}^{b} \int_{c}^{d} \sqrt{1+f_{x}(x, y)^{2}+f_{y}(x, y)^{2}} d x d y
$$

For a general region $R$, the area of a surface over $R$ is

$$
\iint_{R} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^{2}+\left(\frac{\partial z}{\partial y}\right)^{2}} d A
$$

... so any method of evaluating this double integral (type I, type II, polar) is fair game!

## Surface Area

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$$

Find the surface area of ...

1. The part of the plane $5 x+3 y-z+6=0$ above the rectangle $[1,4] \times[2,6]$
2. The part of the surface $2 y+4 z=x^{2}=5$ above the triangle with vertices $(0,0),(2,0)$ and $(2,4)$
3. The part of the hyperbolic paraboloid $z=y^{2}-x^{2}$ that lies between the cylinders $x^{2}+y^{2}=1$ and $x^{2}+y^{2}=4$
4. The part of the sphere $x^{2}+y^{2}+z^{2}=4$ that lies above the plane $z=1$
