# Math 213 - Change of Variables in Double Integrals

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University of Kentucky

November 7, 2018

# Homework

- Re-read section 15.9
- Work on section 15.9, problems 1-37 (odd) from Stewart
- Prepare for Thursday's quiz on sections 15.6-15.8 (triple integrals, coordinates)
- Finish webwork C3 for tonight!

# Unit III: Multiple Integrals

# Goals of the Day

- Understand what a transformation $T$ between two regions in the plane is
- Understand how to compute the *Jacobian Matrix* and *Jacobian determinant* of a transformation and understand what the Jacobian determinant measures
- Understand how to compute double integrals using the change of variables formula

# Preview: Calculus I versus Calculus III

If $x = g(u)$ maps $[c, d]$ to $[a, b]$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du$$

In other words,

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du$$

---

If $x = g(u, v)$, $y = h(u, v)$, and if the region $S$ in the $uv$ plane is mapped to the region $R$ in the $xy$ plane, then

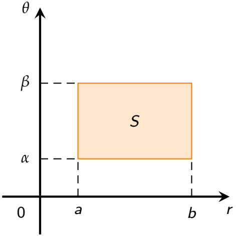
$$\iint_R f(x, y)\,dA = \iint_S f(x(u, v), y(u, v))\,J(u, v)\,du\,dv$$

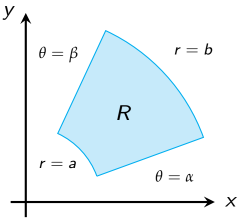
The *Jacobian determinant*

$$J(u, v) = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|$$

measures how areas change under the map $(u, v) \mapsto (x, y)$.

# Transformations




The polar coordinate map

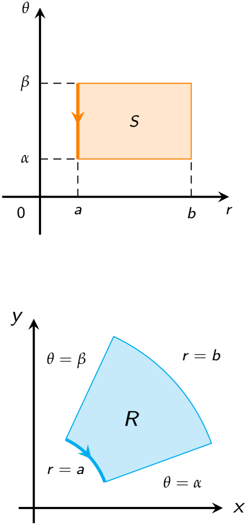
$$x = r\cos\theta, \quad y = r\sin\theta$$

defines a transformation $T : S \to R$

Before, we called $R$ a *polar rectangle*

Here are the corresponding sides of the rectangle in the $r\theta$ plane and the polar rectangle in the $xy$ plane:

# Transformations

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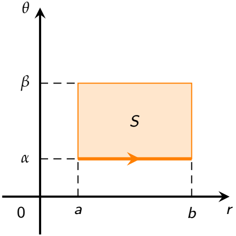


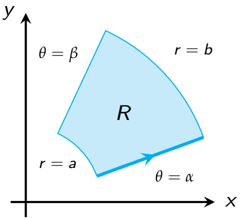


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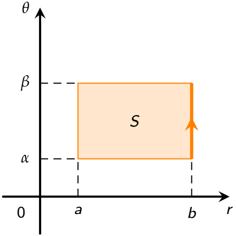


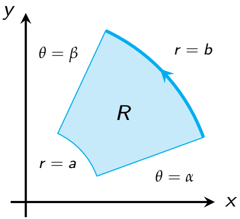


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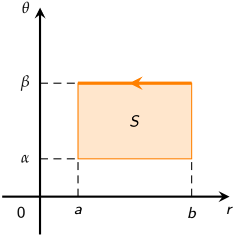


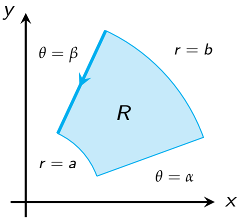


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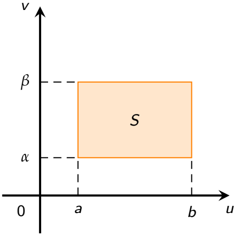


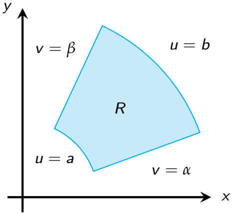


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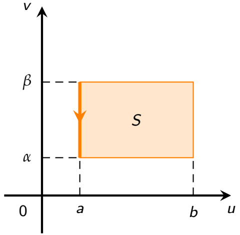


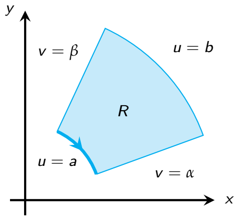


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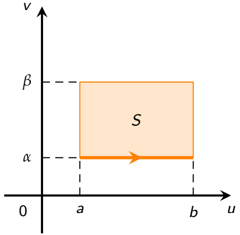


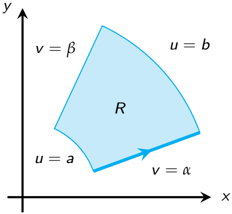


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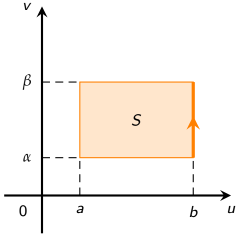


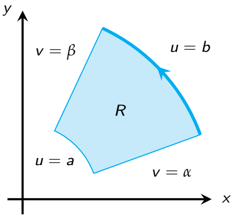


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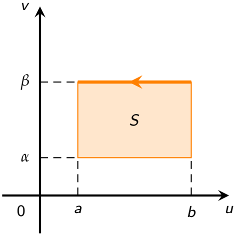


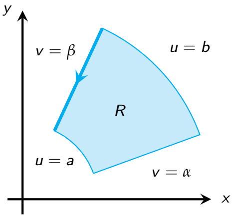


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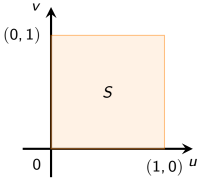
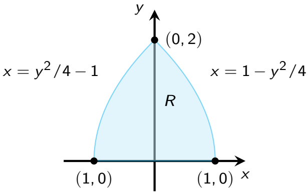



The equations

$$x = u^2 - v^2, \quad y = 2uv$$

defines a transformation $T : S \to R$



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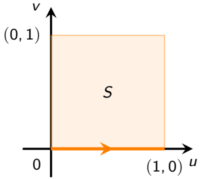


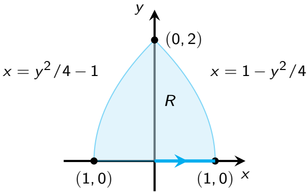


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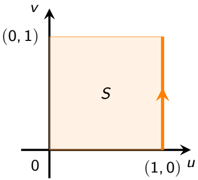


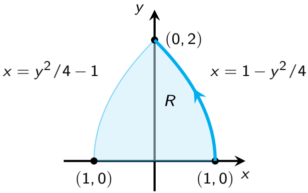


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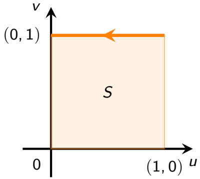


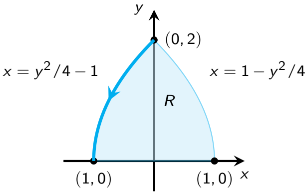


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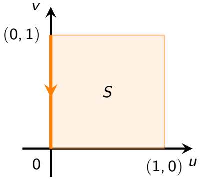


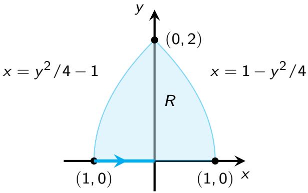


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- $v = 1$, $0 \le u \le 1$ maps to the parametric curve $x = u^2 - 1$, $y = 2u$
- $u = 0$, $0 \le v \le 1$ maps to $-1 \le x \le 0$, $y = 0$

# Transformations

1. Find the image of

$$S = \{(u, v) : 0 \leq u \leq 3,\ 0 \leq v \leq 2\}$$

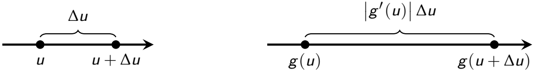
   under the transformation $x = 2u + 3v$, $y = u - v$

2. Find the image of the disc $u^2 + v^2 \leq 1$ under the transformation $x = au$, $y = bv$

# The Jacobian

In *one variable calculus*, the way a transformation $x = g(u)$ changes lengths of interals is measured by $g'(u)$:

$$\Delta x = g'(u)\Delta u$$




In *two variable calculus*, the way a transformation

$$x = g(u, v), \quad y = h(u, v)$$

changes areas is measured by the *Jacobian determinant*

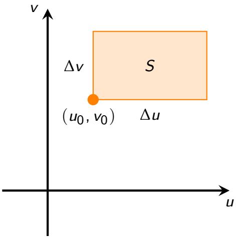
$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial v} \end{vmatrix}, \quad \Delta A = |J(u, v)|\, \Delta u\, \Delta v$$

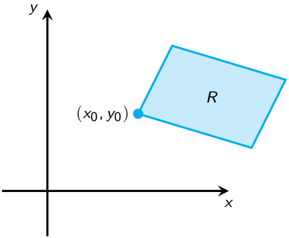
We'll now see why this is the case...

# The Jacobian

A transformation $x = g(u, v)$, $y = h(u, v)$ maps a small rectangle $S$ into a distorted rectangle $R$ through the rule

$$\mathbf{r}(u, v) = g(u, v)\mathbf{i} + h(u, v)\mathbf{j}$$



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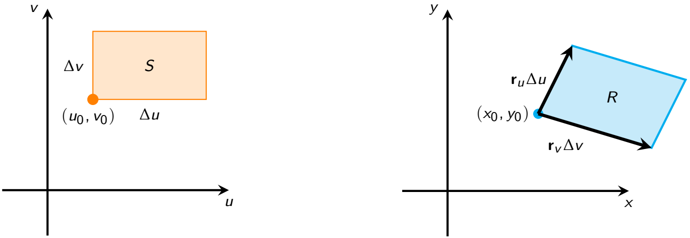
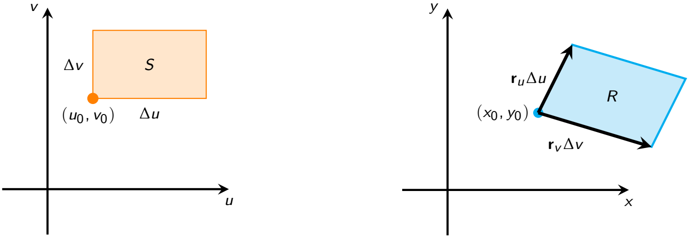



$R$ has approximate area $|\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v$ where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

# The Jacobian

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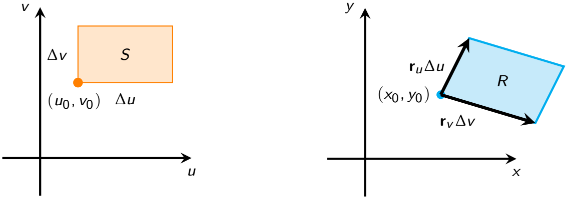
$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}, \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}$$

Compute

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

# The Jacobian




The area of $R$ is approximately

$$dA \simeq \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \, \Delta v$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

is the *Jacobian determinant* of the transformation

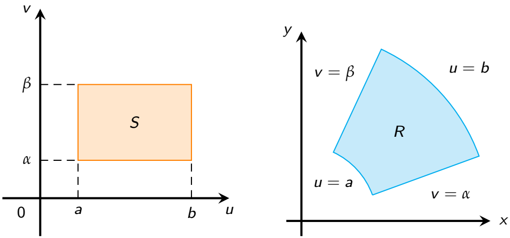
# The Jacobian

Find the Jacobian determinant of the following transformations.

1. $x = 2u + 3v$, $y = u - v$
2. $x = au$, $y = bv$
3. $x = u^2 - v^2$, $y = 2uv$

# Area Change in Polar Coordinates

Consider the transformation $x = u \cos v, \quad y = u \sin v$




$$\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} cos(v) & -u\sin(v) \\ \sin(v) & u\cos(v) \end{pmatrix}$$

so

$$J(u, v) = \begin{vmatrix} cos(v) & \sin(v) \\ -u\sin(v) & u\cos(v) \end{vmatrix} = u$$

# Change of Variables Formula

If the transformation $x = g(u, v)$, $y = h(u, v)$ maps a region $S$ in the $uv$-plane to a region $R$ in the $xy$ plane:

$$\iint_R f(x, y)\, dA \simeq \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A$$

$$\simeq \sum_{i=1}^{n} \sum_{j=1}^{n} f(g(u_i, v_j), h(u_i, v_j)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v$$

$$\iint_S f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

**Change of Variables in a Double Integral** If $T$ is a one-to-one transformation with nonzero Jacobian and $T : S \to R$, then

$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

# Change of Variables Formula

**Change of Variables in a Double Integral** If $T$ is a one-to-one transformation with nonzero Jacobian and $T : S \to R$, then

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---

1. Use the transformation $x = 2u + v$, $y = u + 2v$ to find $\iint_R (x - 3y)\, dA$ if $R$ is the triangular region with vertices $(0,0)$, $(2,1)$ and $1,2)$

2. FInd $\iint_R (x+y)e^{x^2-y^2}\, dA$ if $R$ is the rectangle enclosed by $x - y = 0$, $x - y = 2$, $x + y = 0$, and $x + y = 3$.